Alterations

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Introduction

Throughout the talk, k is a field not necessarily of characteristic 0, often algebraically closed.

- A **variety** over k is an integral separated k-scheme of finite type. A **modification** is a proper birational morphism. An **alteration** of integral schemes is a dominant, proper, and generically finite morphism. In particular, a modification is a birational alteration.

Theorem 1 (Hironaka)

Let k be a field of characteristic 0, X a geometrically integral k-variety, and Z a closed subvariety of X. Then there exists a finite sequence of blow-up at nonsingular closed subvarieties

$$X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X,$$

such that X_n is nonsingular and the strict transform of Z is a normal crossings divisor.

Introduction

- The resolution obtained in Hironaka's theorem is birational, i.e., X_n and X are generically same.
- Naturally one hopes if such a result is true in positive characteristic. This question is still open as of now.
- However, if one allows nontrivial function field extensions, then we have the following recent theorem by de Jong–

Theorem 2 (de Jong)

Let X be a k-variety. Then there exists an alteration $\varphi \colon X' \to X$ such that X' is a regular quasi-projective variety. Additionally, if k is perfect then φ can be arranged to be generically étale[a].

[a] A morphism of schemes $f \colon X \to Y$ is called **generically étale** if there is a dense open subset $U \subseteq Y$ such that $f^{-1}(U) \to U$ is étale.

Introduction

- In order for the induction in the proof to work, de Jong's theorem asserts something more–

Theorem 3 (de Jong)

Let X be a k-variety and $Z \subseteq X$ a proper closed subset. There exists an alteration $\varphi \colon X_1 \to X$ along with an open embedding $j \colon X_1 \to \overline{X_1}$ such that

- *$\overline{X_1}$ is a regular projective variety,*
- *the closed subset $j(\varphi^{-1}Z) \cup \overline{X_1} \setminus j(X_1)$ is the support of a strict normal crossings divisor [a] in $\overline{X_1}$.*

If k is perfect then the alteration φ may be chosen to be generically étale.

[a] A **strict normal crossings divisor** on X is an effective Cartier divisor $D \subset X$ such that for every $p \in D$ the local ring $\mathcal{O}_{X,p}$ is regular and there exists a regular system of parameters $x_1, \ldots, x_d \in \mathfrak{m}_p$ and $1 \leqslant r \leqslant d$ such that D is cut out by $x_1 \cdots x_r$ in $\mathcal{O}_{X,p}$.

Introduction

- de Jong's approach involves constructing a "good" fibration of X consisting of nodal curves. This requires the use of alteration.
- Once the variety is in the desired form, it is possible to use induction on the dimension of the fibration's base space.
- This leads to a scenario where the singularities on the variety are mild and the desingularization can be carried out by hand via explicit blow-ups.
- For simplicity of the exposition, we will assume that k is algebraically closed throughout.

Preliminary reductions and observations

- *Replacing X by an alteration.* If $\varphi\colon X' \to X$ is an alteration, then the theorem follows for (X, Z) if it holds for $(X', \varphi^{-1}(Z))$.

(P2) *X is quasi-projective.* Chow's lemma gives a modification $X' \to X$ such that X' is quasi-projective over k. Hence, we may assume X is quasi-projective.

(P3) *X is projective.* Suppose $j\colon X \hookrightarrow \overline{X}$ be an open embedding of X into a projective variety $\overline{X}$. Put $\overline{Z} = j(Z) \cup \overline{X} \setminus X$. It is clear that if $(\overline{X}, \overline{Z})$ satisfies the theorem then (X, Z) satisfies it as well.

(P4) *Z is the support of an effective Cartier divisor.* Replace (X, Z) by $(\mathrm{Bl}_Z X, \mathrm{E}_Z X)$.

- *Enlarging Z.* If $Z' \subseteq X$ is a closed subset containing Z and we can solve the problem for (X, Z') then we can also solve it for (X, Z).

(P5) *X is normal.* We may replace X by its normalization.

Constructing a good fibration

Lemma 4

Suppose the pair (X, Z) satisfies properties P2-P4. There exist a modification $\varphi\colon X' \to X$ and a morphism $f\colon X' \to \mathbb{P}^{d-1}$, $d = \dim X$, such that

1. *There exists a finite subset $S \subset X \setminus Z$ of regular closed points such that $\varphi\colon X' \to X$ is the blow-up $\mathrm{Bl}_S X \to X$.*
2.
    1. *All fibers of f are nonempty and of pure-dimension 1.*
    2. *The smooth locus of f is dense in all fibers of f.*
    3. *Let $Z' = \varphi^{-1}(Z)$, endowed with the induced reduced closed subscheme structure. The morphism $f|_{Z'}$ is finite and generically étale.*
    4. *If X is normal, i.e., if (X, Z) satisfies P5, then we may arrange for at least one closed fiber of f to be smooth. In particular, this implies that f is generically smooth by generic flatness.*

Constructing a good fibration

Lemma 5

Fix a projective variety $Y \subseteq \mathbb{P}^N$ over an algebraically closed field.

- *If* $\dim Y < N - 1$ *then* pr_p *is finite birational for a general point* p.
- *If* $\dim Y = N - 1$ *then* pr_p *is finite generically étale for a general point* p.

- The above lemma is standard. The idea is to look at secant varieties and tangent varieties. Use generic smoothness to get plenty of regular points on Y, and generic flatness to help with étaleness.

Proof of Lemma 4.

- Consider X as a projective variety in $\mathbb{P}^N$. By using the above lemma, we have a finite generically étale morphsim $\pi \colon X \to \mathbb{P}^d$. We can also ensure that $\pi|_Z$ is birational onto its image.
- Let $B \subseteq \mathbb{P}^d$ be the branch locus of π. Then $\mathrm{pr}_p \colon \pi(Z) \to \mathbb{P}^{d-1}$ is generically étale for a general point $p \notin B$ by using the lemma for each irreducible component of $\pi(Z)$.

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- We view this $\mathbb{P}^{d-1}$ as a linear subspace of $\mathbb{P}^d$ not containing p, call it $\mathbb{G}$. This parametrizes all lines through p. Choose any $p \in \mathbb{P}^d \setminus (B \cup \pi(Z))$ and take $S = \pi^{-1}(p)$.
- Then S is contained in the regular locus of X, and also $S \cap Z = \varnothing$.

$$\begin{array}{ccc} X' := \mathrm{Bl}_{\pi^{-1}(p)} X & \longrightarrow & X \\ \downarrow & & \downarrow \pi \\ \mathrm{Bl}_p \mathbb{P}^d & \longrightarrow & \mathbb{P}^d \end{array}$$

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We are using that blow-ups commute with flat base change.

$$\mathrm{Bl}_p \mathbb{P}^d = \{(x, \ell) \in \mathbb{P}^d \times \mathbb{G} : x \in \ell\}, \qquad X' = \{(x, \ell) \in X \times \mathbb{G} : \pi(x) \in \ell\}.$$

- Choose $f : X' \to \mathbb{G} = \mathbb{P}^{d-1}$. The fiber of f over $\ell \in \mathbb{G}$ is $\pi^{-1}(\ell)$. Since ℓ is locally cut out by $d-1$ equations, it follows that $\pi^{-1}(\ell)$ has pure dimension 1. Every irreducible component of a fiber intersects $\pi^{-1}(p)$.
- The second last part is clear as $f|_{Z'} : Z' \cong Z \to \pi(Z) \to \mathbb{P}^{d-1}$ is generically étale (and finite) by construction.

Constructing a good fibration

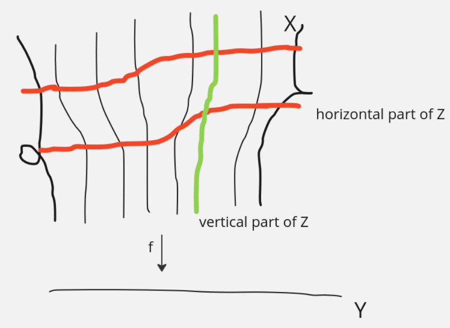
- The last assertion of generic smoothness of $f\colon X \to \mathbb{P}^{d-1}$ comes from iterated Bertini since a fiber of f is obtained by intersecting a $N-d+1$ dimensional linear subspace $H \subseteq \mathbb{P}^N$ containing a (fixed) $N-d$ dimensional linear subspace $L \subseteq \mathbb{P}^N$. The exact details are nontrivial.
- We remark that all fibers of $f\colon X \to \mathbb{P}^{d-1}$ are geometrically connected. This comes from a routine application of Stein factorisation and simply-connectedness of $\mathbb{P}^{d-1}$.

Situation as of now

We now replace (X, Z) with (X', Z') so that we may assume properties P2-P5 along with

(P6) There exists a morphism $f\colon X \to Y$ of projective varieties such that

1. All fibers are nonempty, geometrically connected and of pure dimension 1.
2. The smooth locus of f is dense in all fibers and f is generically smooth.
3. $f|_Z$ is generically étale, ...



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- In particular, f is a generically nodal family of curves. Suppose f is smooth over the open set $U \subset Y$. We would like to extend $f^{-1}(U) \to U$ to a family of nodal curves over whole of X.
- If the moduli functor

$$T \mapsto \{\text{proper family of nodal curves over } T\}_{/\simeq}$$

were representable by a projective scheme then we win by "taking closure".

- The problem is that the the above functor is not representable. Therefore, we want to work with nodal families with n marked sections.

Straightening out Z

- Let $\psi\colon Y' \to Y$ be a generically étale alteration. In the rest of this article, we will frequently make the transformation

$$X' := (X \times_Y Y')_{\mathrm{red}}, \qquad Z' := (Z \times_Y Y')_{\mathrm{red}}, \qquad Y'.$$

It can be verified that this preserves most of the important properties.

Proposition 1

In the above setting, we can choose ψ so that $Z' = \cup_{i=1}^{r} \sigma_i(Y')$ for distinct sections $\sigma_i\colon Y' \to X'$.

Proof.

- Let η be the generic point of Y. From our assumptions, Z_η is a nonempty finite étale η-scheme consisting of generic points of Z as Z is generically étale over Y.
- Choose a finite Galois point $\eta' \to \eta$ so that $Z_\eta \times_\eta \eta'$ is a finite disjoint union of copies of η'.

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- Choose a finite Galois point $\eta' \to \eta$ so that $Z_\eta \times_\eta \eta'$ is a finite disjoint union of copies of η'.
- Take $\psi\colon Y' \to Y$ to be the normalization of Y in the finite Galois extension $\kappa(\eta')/\kappa(\eta)$.
- We relabel and write X, Y, Z, η to mean X', Y', Z', η'.
- Then each finite morphism $Z_i \to Y$ is surjective (because of dimension reasons) and hence, birational too. Indeed, Z_η is precisely the collection of generic points of irreducible components of Z and $\kappa(Z_i) \cong \kappa(Y)$ by construction.

Straightening out Z

Proof.

- Choose a finite Galois point $\eta' \to \eta$ so that $Z_\eta \times_\eta \eta'$ is a finite disjoint union of copies of η'. Indeed, if $Z_i, 1 \leqslant i \leqslant r$, are the irreducible components of Z, then we can choose $\kappa(\eta')$ to be any Galois extension of $\kappa(\eta)$ containing all of $\kappa(Z_i), i \leqslant i \leqslant r$.
- Take $\psi\colon Y' \to Y$ to be the normalization of Y in the finite Galois extension $\kappa(\eta')/\kappa(\eta)$.
- Then each finite morphism $Z_i \to Y$ is surjective (because of dimension reasons) and hence, birational too. Indeed, Z_η is precisely the collection of generic points of irreducible components of Z and $\kappa(Z_i) \cong \kappa(Y)$ by construction.
- As Y is normal, it follows that $Z_i \to Y$ is an isomorphism by Zariski's main theorem. Thus, their inverses $Y \to Z_i$ are the desired sections.

Producing a stable pointed family

- Define

$$U = \{y \in Y : X_y \text{ is smooth over } y \text{ and } \sigma_i(y) \neq \sigma_j(y) \text{ for } i \neq j\} \subset Y.$$

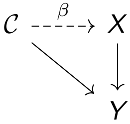
By P6 (c) (generic smoothness of f), it follows that U is a nonempty open set. So, $X_U \to U$ is a family of stable n-pointed curves[1].

- By some moduli space techniques (stable extension theorem) which we don't go into, one can ensure, at least after an alteration of the base, properties P2-P4, P6 (a)-(f) along with

(P6) (g) There exists a family of stable n-pointed curve $(\mathcal{C}, \tau_1, \dots, \tau_n)$ over Y, a nonempty open subscheme $U \subset Y$, and an U-isomorphism $\beta : \mathcal{C}_U \to X_U$ mapping the sections $\tau_i|_U$ to $\sigma_i|_U$.

[1] A family $\mathcal{C} \to S$ of nodal curves together with sections $\sigma_i : S \to \mathcal{C}$, $i = 1, \dots, n$, is called a **family of stable n-pointed curves of genus** g if (i) $\sigma_i(S)$ lie in the smooth locus $(\mathcal{C}/S)^{\text{sm}}$ and are mutually disjoint, (ii) All geometric fibers have arithmetic genus g, and (iii) $\omega_{\mathcal{C}/S}(\sum \sigma_i(S))$ is relatively ample.

Extending β



- We can base-change the diagram above to the normalization of Y and we do so.
- Ideally, we want β to extend to a regular map because then we can replace X by $\mathcal{C}$.
- A common technique to extend a rational map is to pass to the closure of the graph. Define T as the closure of the graph $\Gamma_\beta \subset \mathcal{C} \times_Y X$.
- Then β is a regular map if and only if $\mathrm{pr}_1 \colon T = \overline{\Gamma}_\beta \to \mathcal{C}$ is an isomorphism.

Extending β

The induced map $T \to Y$ may not have curve as fibers, so we flatten it–

Theorem 6 (Raynaud-Gruson)

Let X and Z be varieties over a perfect field and $X \to Z$ a dominant projective morphism. There exists a modification $f : Y \to Z$ such that the strict transform $f' : \widetilde{X_Y} \to Y$ is flat.

- So we blow up Y and assume that X and T are Y-flat.
- We already know that pr_1 is birational because β is an isomorphism over U. Also, as Y is normal, $\mathcal{C}$ is normal (this comes from Serre's $R_1 + S_2$ criterion). So we hope to apply Zariski's main theorem.
- So we wish to show that pr_1 has finite fibers.
- And now we come at a very technical discussion, which I am not going to pursue here. So let us just assume that pr_1 is magically an isomorphism and consequently β extends to a morphism.

Blow-ups

- Finally, we replace (X, Z) by $(\mathcal{C}, \beta^{-1}(Z))$. Here, we may lose the finiteness of $Z \to Y$ but that's a non-issue.
- One can then use induction on dimension to change Y to a regular scheme.
- The resulting X has very simple singularities, and its desingularization can be carried out by hand.
- Indeed, a generically smooth family of nodal curves looks, étale locally around a singularity, something like

$$\operatorname{Spec} \frac{k[x, y, t]}{(xy - t^2)} \to \operatorname{Spec} k[t],$$

which can be resolved by routine blow-ups at singularities.

References

D. Abramovich and F. Oort, *Alterations and Resolution of singularities*, Resolution of Singularities, Progress in Mathematics **181**, Birkhäuser, Basel.

T. Feng, and A. Landesman, *Math 249B Notes: Alterations*, lectures by B. Conrad, http://math.stanford.edu/~conrad/249BW17Page/handouts/alterations-notes.pdf.

The Stacks project authors, *The Stacks project*, 2023.

A. J. de Jong, *Smoothness, semi-stability and alterations*. Publications Mathématiques IHÉS **83** (1996), pp. 51-93.