RESOLUTION OF SINGULARITIES IN ARBITRARY CHARACTERISTIC

AYAN NATH

Contents

Abstract. The primary aim of this report is to provide an account of de Jong's theorem on alterations [dJ], focusing on its principal arguments. Additionally, we also discuss the resolution of singularities of curves that are embedded in varieties, with no assumptions on characteristic. Preliminary facts on blow-ups and a few examples are presented in an appendix.

1 Alterations

Most of the material in this section is taken from [dJ], [AbOo], and [AltNotes].

1.1. Definitions. A **variety** over k is an integral separated k-scheme of finite type. A **modification** is a proper birational morphism. An **alteration** of integral schemes is a dominant, proper, and generically finite morphism. In particular, a modification is a birational alteration.

We recall the celebrated theorem of Hironaka–

1.2. Hironaka's Theorem. — *Let k be a field of characteristic 0, X a geometrically integral k-variety, and Z a closed subvariety of X. Then there exists a finite sequence of blow-up at nonsingular closed subvarieties*

$$X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X,$$

such that X_n is nonsingular and the strict transform of Z is a normal crossings divisor.

Date: 4th May, 2023.

Affiliation: BSc 2nd year, Chennai Mathematical Institute.

The resolution obtained in Hironaka's theorem is birational, i.e., X_n and X are generically same. Naturally one hopes if such a result is true in positive characteristic. This question is still open as of now. However, if one allows nontrivial function field extensions, then we have the following recent theorem by de Jong–

1.3. de Jong's Theorem. — *Let X be a k-variety. Then there exists an alteration $\varphi\colon X' \to X$ such that X' is a regular quasi-projective variety. Additionally, if k is perfect then φ can be arranged to be generically étale*[1].

In order for the induction in the proof to work, de Jong's theorem asserts something more–

1.4. de Jong's Theorem*. — *Let X be a k-variety and $Z \subseteq X$ a proper closed subset. There exists an alteration $\varphi\colon X_1 \to X$ along with an open embedding $j\colon X_1 \to \overline{X_1}$ such that*

- *$\overline{X_1}$ is a regular projective variety,*
- *the closed subset $j(\varphi^{-1}Z) \cup \overline{X_1} \setminus j(X_1)$ is the support of a strict normal crossings divisor in $\overline{X_1}$.*

If k is perfect then the alteration φ may be chosen to be generically étale.

de Jong's approach involves constructing a "good" fibration of X consisting of nodal curves. This requires the use of alteration. Once the variety is in the desired form, it is possible to use induction on the dimension of the fibration's base space. This leads to a scenario where the singularities on the variety are quite mild and the desingularization can be carried out by hand.

For simplicity of the exposition, we will assume that k is algebraically closed throughout. Further, all alterations considered are generically étale.

1.5. Preliminary reductions and observations.

- *Replacing X by an alteration.* If $\varphi\colon X' \to X$ is an alteration, then the theorem follows for (X,Z) if it holds for $(X', \varphi^{-1}(Z))$.
- (P2) *X is quasi-projective.* Chow's lemma gives a modification $X' \to X$ such that X' is quasi-projective over k. Hence, we may assume X is quasi-projective.
- (P3) *X is projective.* Suppose $j\colon X \hookrightarrow \overline{X}$ be an open embedding of X into a projective variety $\overline{X}$. Put $\overline{Z} = j(Z) \cup \overline{X} \setminus X$. It is clear that if $(\overline{X}, \overline{Z})$ satisfies the theorem then (X,Z) satisfies it as well.
- (P4) *Z is the support of an effective Cartier divisor.* Replace (X,Z) by $(\mathrm{Bl}_Z X, \mathrm{E}_Z X)$.
- *Enlarging Z.* If $Z' \subseteq X$ is a closed subset containing Z and we can solve the problem for (X,Z') then we can also solve it for (X,Z).
- (P5) *X is normal.* We may replace X by its normalization.

1.6. Constructing a good fibration. Denote $d = \dim X$. The goal is to prove the following

1.7. Lemma. — *Suppose the pair (X,Z) satisfies properties P2-P4. There exist a modification $\varphi\colon X' \to X$ and a morphism $f\colon X' \to \mathbb{P}^{d-1}$ such that*

(i) There exists a finite subset $S \subset X \setminus Z$ of regular closed points such that $\varphi\colon X' \to X$ is the blow-up $\mathrm{Bl}_S X \to X$.

(ii) (a) All fibers of f are nonempty and of pure-dimension 1.

(b) The smooth locus of f is dense in all fibers of f.

(c) Let $Z' = \varphi^{-1}(Z)$, endowed with the induced reduced closed subscheme structure. The morphism $f|_{Z'}$ is finite and generically étale.

(d) If X is normal, i.e., if (X,Z) satisfies P5, then we may arrange for at least one closed fiber of f to be smooth. In particular, this implies that f is generically smooth by generic flatness.

[1]A morphism of schemes $f\colon X \to Y$ is called **generically étale** if there is a dense open subset $U \subseteq Y$ such that $f^{-1}(U) \to U$ is étale.

We begin with a few basic lemmas. Fix a projective variety $Y \subseteq \mathbb{P}^N$ over an algebraically closed field.

1.8. Lemma. — *Suppose* $\dim Y < N-1$. *Then there is a nonempty open subset* $U \subseteq \mathbb{P}^N$ *such that if* $p \in U$ *then the projection* $\mathrm{pr}_p \colon Y \to \mathbb{P}^{N-1}$ *is finite birational onto its image.*

Proof. Standard. Use generic smoothness and see [Har77, Proposition IV.3.5] for the general idea. □

1.9. Lemma. — *Suppose* $\dim Y = N-1$. *Then there is a nonempty open subset* $U \subset \mathbb{P}^N$ *such that if* $p \in U$ *then the projection* $\mathrm{pr}_p \colon Y \to \mathbb{P}^{N-1}$ *is finite generically étale.*

Proof. We have plenty of regular points by generic smoothness. Imitating the proof of Lemma 1.8, except that we consider tangent varieties instead of secant varieties, we obtain that pr_p is unramified for a general point p. In this setting, unramified implies generically étale due to generic flatness. □

1.10. *Proof of Lemma 1.7.* Consider X as a projective variety in $\mathbb{P}^N$. By using Lemmas 1.8 and 1.9, we have a finite generically étale morphsim $\pi\colon X \to \mathbb{P}^d$. By Lemma 1.8 we can also ensure that $\pi|_Z$ is birational onto its image. Let $B \subseteq \mathbb{P}^d$ be the branch locus of π. Then $\mathrm{pr}_p \colon \pi(Z) \to \mathbb{P}^{d-1}$ is generically étale for a general point $p \notin B \cup \pi(Z)$. Indeed, we may apply Lemma 1.9 to each irreducible component of $\pi(Z)$. We view this $\mathbb{P}^{d-1}$ as a linear subspace of $\mathbb{P}^d$ not containing p. To distinguish this from the usual projective space, we write $\mathbb{G}$ instead of $\mathbb{P}^{d-1}$. Note that $\mathbb{G}$ parametrizes all lines in $\mathbb{P}^d$ passing through p so we may identify $\mathbb{G}$ with the space of all lines through p. Choose any $p \in \mathbb{P}^d \setminus (B \cup \pi(Z))$ and take $S = \pi^{-1}(p)$. By definition of branch locus, S is contained in the regular locus of X, and also $S \cap Z = \varnothing$. Set

$$X' = \{(x,\ell) \in X \times \mathbb{G} \colon \pi(x) \in \ell\}.$$

We claim that $X' \cong \mathrm{Bl}_S X$. Indeed, we have the following fibered diagram–

$$\begin{array}{ccc} X' & \longrightarrow & \mathrm{Bl}_p \mathbb{P}^d \\ \downarrow & & \downarrow \\ X & \xrightarrow[\pi]{} & \mathbb{P}^d \end{array}$$

Because π is étale, in particular flat, over an open set containing p and blow-ups commute with flat base change we get $X' \cong \mathrm{Bl}_S X$. Consider $f = \mathrm{pr}_2 \colon X' \to \mathbb{G} = \mathbb{P}^{d-1}$. We claim that this is the required morphism. The fiber of f over $\ell \in \mathbb{G}$ is $\pi^{-1}(\ell)$. As ℓ is a line and π is finite, it follows that $\pi^{-1}(\ell)$ has dimension at most 1. Furthermore, ℓ is given by $d-1$ equations locally, hence $\pi^{-1}(\ell)$ has pure dimension 1. Also, by our very construction, every irreducible component of $\pi^{-1}(\ell)$ contains at least one point of $\pi^{-1}(p)$ because $p \in \ell$ and $\pi^{-1}(\ell) \to \ell$ is finite, and in particular, has finite fibers. Further, the smooth locus $(X'/\mathbb{P}^{d-1})^{\mathrm{sm}}$ is open. This completes the proofs of (i), (ii) (a)-(b). Assertion (ii) (c) is clear as $f|_{Z'} \colon Z' \cong Z \to \pi(Z) \to \mathbb{P}^{d-1}$ is generically étale (and finite) by construction.

The last assertion comes from iterated Bertini since a fiber of f is obtained by intersecting a $N-d+1$ dimensional linear subspace $H \subseteq \mathbb{P}^N$ containing a (fixed) $N-d$ dimensional linear subspace $L \subseteq \mathbb{P}^N$. The exact details are nontrivial. See `http://math.stanford.edu/~conrad/249BW17Page/handouts/genericity.pdf`. □

1.11. Lemma. — *All fibers of f are geometrically connected.*

Proof. Put $S = (X'/\mathbb{P}^d)^{\mathrm{sm}} \subset X'$, the smooth locus of f. Let $S \xrightarrow{f'} Z \xrightarrow{g} \mathbb{P}^d$ be the Stein factorisation of $S \to \mathbb{P}^d$. Since smooth morphisms are flat and $f'_*\mathcal{O}_S = \mathcal{O}_Z$, it follows that g is flat. I claim that g is unramified. Indeed, if $p \in \mathbb{P}^d$ and the finite set Z_p has points with multiplicity at least two, i.e., nonreduced points, then the fiber S_p would be also nonreduced because some components will occur with multiplicity at least 2. But this is not possible because S_p is p-smooth. We conclude that g is étale and hence g is a trivial cover due to simply-connectedness of $\mathbb{P}^d$. Thus, $S \to \mathbb{P}^d$ has geometrically connected fibers. As S is fiberwise-dense in $X/\mathbb{P}^d$, this completes the proof. $\square$

We now replace (X, Z) with (X', Z') so that we may assume properties P2-P5 along with

(P6) There exists a morphism $f\colon X \to Y$ of projective varieties such that
  (a) All fibers are nonempty, geometrically connected and of pure dimension 1.
  (b) The smooth locus of f is dense in all fibers.
  (c) f is generically smooth.
  (d) $f|_Z\colon Z \to Y$ is finite and generically étale.

In the sequel, we will relax the property "all fibers are geometrically connected" to "general fibers are geometrically connected".

1.12. Enlarging the divisor. Consider the following lemma

1.13. Lemma. — *Let $f\colon X \to Y$ be as above, satisfying P6 (a) and (b). There exists an effective Cartier divisor $H \subset X$ such that*

(1) $f|_H\colon H \to Y$ is finite and generically étale,
(2) for any irreducible component C of a geometric fiber of f, we have

$$\#(X/Y)^{\mathrm{sm}} \cap C \cap H \geqslant 3,$$

counted without multiplicities.

Proof. Fix a large natural number n and a very ample line bundle $\mathscr{L}$ on X. Let $i\colon X \hookrightarrow \mathbb{P}$ be the closed embedding associated to $\mathscr{L}^{\otimes n}$. For any irreducible curve $C \subset X$, the image $i(C) \subset \mathbb{P}$ is not contained in any linear subspace of dimension $n-1$. This is because i is the closed embedding given by $\mathscr{L}$ followed by an n-uple embedding.

Denote by $\mathbb{P}^\vee$ the dual projective space of all hyperplanes in $\mathbb{P}$. Define the incidence variety

$$T := \{(H, y) \in \mathbb{P}^\vee \times Y\colon \dim f^{-1}(y) \cap H = 1\} \subset \mathbb{P}^\vee \times Y.$$

This is clearly a closed set. Indeed, it is the locus where

$$\{(H, x) \in \mathbb{P}^\vee \times X\colon x \in H\} \xrightarrow{\mathrm{id}_{\mathbb{P}^\vee} \times f} \mathbb{P}^\vee \times Y$$

has fibers of dimension 1; so by upper-semicontinuity of fiber dimension and that fibers of f have dimension at most 1, it follows. Let $\overline{y}$ be a geometric point of Y. Then the geometric fiber $\mathrm{pr}_2^{-1}(\overline{y})$ is

$$\bigcup_{\substack{C \subset f^{-1}(\overline{y}) \\ \text{irreducible component}}} \{H \subset \mathbb{P}^\vee \times_k \kappa(\overline{y})\colon i(C) \subset H\}.$$

As $i(C)$ is not contained in any linear subspace of dimension $n-1$, it follows that $\operatorname{codim}_{\mathbb{P}^\vee \times_k \kappa(\overline{y})} \mathrm{pr}_2^{-1}(\overline{y}) \geqslant n$. Therefore, $\dim T \leqslant \dim Y + \dim \mathbb{P}^\vee - n$. Choose n so large that $\mathrm{pr}_1(T)$ has positive codimension in $\mathbb{P}^\vee$. Fix a closed point $y \in Y(k)$. We claim that the conditions

(i) $H \notin \mathrm{pr}_1(T)$,
(ii) $H \cap f^{-1}(y) \subset (X/Y)^{\mathrm{sm}}$,
(iii) H intersects $f^{-1}(y)$ transversally.

are all generic. Indeed, (iii) is obviously generic (c.f. [Har77, Exercise IV.3.9]) and we have already chosen n so that (i) is generic. Condition (ii) is generic because $H \cap f^{-1}(y) \subset (X/Y)^{\mathrm{sm}}$ is same as saying that H avoids all the (finitely many) singular points of $f^{-1}(y)$. Take any H' satisfying the above conditions and put $H = X \cap H'$. It's clear that $f|_H\colon H \to Y$ is finite due to quasi-finiteness. Note that $f|_H$ has a smooth fiber above y because $H \cap f^{-1}(y) \subset (X/Y)^{\mathrm{sm}}$ Hence $(f|_H)_y$ is étale over $y = \operatorname{Spec} k$ because of dimension reasons. Since k is algebraically closed, $(f|_H)_y$ must therefore be a finite disjoint union of copies of $\operatorname{Spec} k$. Fix any closed point $x \in H \cap f^{-1}(y)$. By smoothness, $\widehat{\mathcal{O}_{X,x}} \cong \widehat{\mathcal{O}_{Y,y}}[[t]]$. The local equation h of H in $\widehat{\mathcal{O}_{X,x}}$ must therefore satisfy $h \in k^\times \cdot t + \mathfrak{m}_y \widehat{\mathcal{O}_{X,x}}$ because $\widehat{\mathcal{O}_{f^{-1}(y),x}} \cong \widehat{\mathcal{O}_{X,x}}/\mathfrak{m}_y \widehat{\mathcal{O}_{X,x}} \cong k[[t]]$ and $\mathcal{O}_{H \cap f^{-1}(y),x} \cong k$. Thus, $\widehat{\mathcal{O}_{X \cap H,x}} \cong \widehat{\mathcal{O}_{X,x}}/(h) \cong \widehat{\mathcal{O}_{Y,y}}[[t]]/(h) \cong \widehat{\mathcal{O}_{Y,y}}$ and we conclude that $f|_H$ is étale in a neighborhood of y, i.e., $f|_H$ is generically étale.

The preceeding discussion shows that for every closed point $y \in Y(k)$, we get a corresponding open $U_p \subset \mathbb{P}^\vee$ satisfying (i)-(iii). Take any $H \in U_p$. For a different closed point $q \in Y(k)$, if $H' \in U_q$ is a hyperplane satisfying (i)-(iii), then the divisor $H + H'$ satisfies conditions (i)-(iii) over $U_p \cup U_q$. Indeed, H and H' being generically étale over a reduced scheme Y means that both H and H' are generically reduced, hence, the divisor $H + H'$, cut out by the (invertible) ideal sheaf $\mathscr{I}_{H/X} \cdot \mathscr{I}_{H'/X}$, coincides with the scheme $(H \cup H')_{\mathrm{red}}$ on a dense open set. Consequently, $H + H' \to Y$ is also generically étale. Since $\mathbb{P}^\vee$ is Noetherian, which in particular implies that any subset is quasicompact, the proof is complete. □

Now, replace (X, Z) with $(X, Z \cup H)$ and further assume

(P6) (e) For all geometric points $\overline{y}$ of Y and any irreducible component C of $X_{\overline{y}}$ we have

$$\#(X/Y)^{\mathrm{sm}} \cap C \cap Z \geqslant 3.$$

1.14. Altering the base. Let $\psi \colon Y' \to Y$ be a generically étale alteration. In the rest of this article, we will frequently make the transformation

$$\begin{aligned} X' &:= (X \times_Y Y')_{\mathrm{red}} \\ Z' &:= (Z \times_Y Y')_{\mathrm{red}} \\ Y'&. \end{aligned}$$

These satisfy all of our conditions except possibly P5. Let f be smooth with geometrically connected fibers of dimension 1 over $U \subset Y$, and $U' = \psi^{-1}(U)$. Then $(X \times_Y Y')_{U'}$ is smooth over U', and hence reduced. Therefore, $X'_{U'} \to U'$ is smooth and has geometrically connected fibers of dimension 1. It's clear that Z' is the support of an effective Cartier divisor of X' once we prove the following lemma–

1.15. Lemma. — *X' is an integral scheme.*

Proof. Of course, X' is reduced by construction. Note that $(X/Y)^{\mathrm{sm}} \times_Y Y'$, being Y'-smooth, is reduced and contained in X'. Therefore, the smooth locus $(X'/Y')^{\mathrm{sm}} \supseteq (X/Y)^{\mathrm{sm}} \times_Y Y'$ is fiberwise-dense over Y' because the same is true for $(X/Y)^{\mathrm{sm}}$ over Y. It suffices to show that $(X'/Y')^{\mathrm{sm}}$, being dense, is irreducible. As $\pi \colon (X'/Y')^{\mathrm{sm}} \to Y'$ is flat, the generic fiber π_η consists exactly of generic points of $(X'/Y')^{\mathrm{sm}}$. However, this generic fiber is irreducible since it is η-smooth and connected. The last step follows from $X \to Y$ being generically smooth and having *geometrically* connected general fibers. Thus, $(X'/Y')^{\mathrm{sm}}$ is irreducible and the claim is proved. □

We additionally assume Y is normal by taking ψ to be a normalization morphism.

1.16. Passage to a union of sections.

1.17. Lemma. — *In the above settings, we can choose ψ so that*

$$Z' = \cup_{i=1}^r \sigma_i(Y')$$

for distinct sections $\sigma_i \colon Y' \to X'$.

Proof. Let $\eta = \operatorname{Spec} \kappa(Y)$ be the generic point of Y. From our assumptions, Z_η is a nonempty finite étale η-scheme consisting of generic points of Z as Z is generically étale over Y. Choose a finite Galois point $\eta' \to \eta$, i.e., $\kappa(\eta) \hookrightarrow \kappa(\eta')$ is a finite Galois extension, so that $Z_\eta \times_\eta \eta'$ is a finite disjoint union of copies of η'. Indeed, if $Z_1, Z_2, \ldots, Z_r$ are the irreducible components of Z, then we can choose $\kappa(\eta')$ to be any Galois extension of $\kappa(\eta)$ containing the residue field at any maximal ideal of the $\kappa(Y)$-algebra $\kappa(Z_1)\otimes_{\kappa(Y)}\kappa(Z_2)\otimes_{\kappa(Y)}\cdots\otimes_{\kappa(Y)}\kappa(Z_r)$. Take $\psi\colon Y'\to Y$ to be the normalization of Y in the finite Galois extension $\kappa(\eta')/\kappa(\eta)$. We relabel and write X,Y,Z,η to mean X',Y',Z',η'. We conclude that, each finite morphism $Z_i\to Y$ is surjective (because of dimension reasons) and hence, birational too. Indeed, Z_η is precisely the collection of generic points of irreducible components of Z and $\kappa(Z_i)\cong\kappa(Y)$ by construction. As Y is normal, it follows that $Z_i\to Y$ is an isomorphism by Zariski's main theorem. Thus, their inverses $Y\to Z_i$ are the desired sections. □

1.18. Producing a family of stable pointed curves. Define

$$U=\{y\in Y\colon X_y \text{ is smooth over } y \text{ and } \sigma_i(y)\neq\sigma_j(y) \text{ for } i\neq j\}\subset Y.$$

By P6 (c), it follows that U is a nonempty open set. So, $X_U\to U$ is a family of stable n-pointed curves. By some deep moduli space techniques which we don't go into, one can ensure, at least after an alteration of the base, properties P2-P4, P6 (a)-(f) along with

(P6) (g) There exists a family of stable n-pointed curve[2] if $(\mathscr{C},\tau_1,\dots,\tau_n)$ over Y, a nonempty open subscheme $U\subset Y$, and an U-isomorphism $\beta\colon\mathscr{C}_U\to X_U$ mapping the sections $\tau_i|_U$ to $\sigma_i|_U$.

One can verify that P6 (g) is stable under generically étale alterations of Y. Ideally, we want β to extend to a regular map. A common technique to extend a rational map is to pass to the closure of the graph. Define T as the scheme-theoretic closure of the graph $\Gamma_\beta\subset\mathscr{C}\times_Y X$. We remark that T is integral, being the closure of $\mathscr{C}_U\cong X_U$ (Lemma 1.15).

1.19. The flatenning lemma (Raynaud-Gruson). — *Let X and Z be varieties over a perfect field and $X\to Z$ a dominant projective morphism. There exists a modification $f\colon Y\to Z$ such that the strict transform $f'\colon\tilde{X}_Y\to Y$ is flat.*

By the above lemma, we can assume, in addition to P6 (a)-(c), (e), and (g), that

(P6) (h) X and T are Y-flat.

We can further normalize Y to assume Y is normal and we do so.

1.20. Lemma. — *$\mathscr{C}$ is normal.*

Proof. We use Serre's R_1+S_2 criterion for normality. Let $p\mapsto q$ under $\mathscr{C}\to Y$. By flatness,

$$\dim\mathscr{O}_{\mathscr{C},p}=\dim\mathscr{O}_{Y,q}+\dim\mathscr{O}_{\mathscr{C}_q,p}.$$

We wish to show that $\mathscr{O}_{\mathscr{C},p}$ is regular when its dimension is at most 1, and $\mathscr{O}_{\mathscr{C},p}$ has a regular sequence of length 2 otherwise. If $\dim\mathscr{O}_{Y,q}\geqslant 2$ then $\mathscr{O}_{\mathscr{C}_q,p}$ has a regular sequence of length 2 by the same criterion as Y is normal. Flat pullback of a regular sequence is regular. So let us assume $\dim\mathscr{O}_{Y,q}\leqslant 1$. As $\mathscr{C}$ is generically smooth by P6 (c), we may assume $\dim\mathscr{O}_{Y,q}=1$, i.e., $\mathscr{O}_{Y,q}$ is a DVR. Here we are using normality of Y. Let π be a uniformizer for $\mathscr{O}_{Y,q}$. We have $\mathscr{O}_{\mathscr{C},p}/(\pi)\cong\mathscr{O}_{\mathscr{C}_q,p}$. Since $\mathscr{C}_q$ is a curve, p is either a generic point of $\mathscr{C}_q$ or a closed point. If p is a generic point then we are done. Else, we seek a nonunit nonzerodivisor in $\mathscr{O}_{\mathscr{C}_q,p}$. Such an element exists because the set of zero divisors in a Noetherian *reduced* ring is exactly the union of its minimal primes. □

If $\mathrm{pr}_1\colon T\to\mathscr{C}$ were quasi-finite (and hence finite) then it would be a finite birational map from a variety to a normal variety which, by Zariski's main theorem, is an isomorphism.

[2]An S-scheme $\mathscr{C}$ is called a **family of nodal curves** if it is of finite presentation, proper and flat, and all geometric fibers are connected reduced curves with at most nodes as singularities. A family $\mathscr{C}\to S$ of nodal curves together with sections $\sigma_i\colon S\to\mathscr{C}$, $i=1,\dots,n$, is called a **family of stable n-pointed curves of genus** g if (i) $\sigma_i(S)$ lie in the smooth locus $(\mathscr{C}/S)^{\mathrm{sm}}$ and are mutually disjoint, (ii) All geometric fibers have arithmetic genus g, and (iii) $\omega_{\mathscr{C}/S}(\sum\sigma_i(S))$ is relatively ample.

1.21. Extending β. Our goal is to show that pr_1 has finite fibers. Such a property can be checked at geometric points. So let $\overline{y} \in Y$ be a geometric point. Observe that each geometric fiber $T_{\overline{y}}$ is a curve due to flatness. We want to show that $(\mathrm{pr}_1)_{\overline{y}}\colon T_{\overline{y}} \to \mathscr{C}_{\overline{y}}$ has finite fibers. We have the following setup–

$$\begin{tikzcd} & T_{\overline{y}} \arrow[dl, "(\mathrm{pr}_1)_{\overline{y}}"'] \arrow[dr, "(\mathrm{pr}_2)_{\overline{y}}"] \arrow[dd, "h_{\overline{y}}"] & \\ \mathscr{C}_{\overline{y}} \arrow[dr] & & X_{\overline{y}} \arrow[dl] \\ & \overline{y} & \end{tikzcd}$$

Splitting into irreducible components–

$$\begin{tikzcd} & T_{\overline{y}} = T_1 \cup \cdots \cup T_t \arrow[dl, "(\mathrm{pr}_1)_{\overline{y}}"'] \arrow[dr, "(\mathrm{pr}_2)_{\overline{y}}"] & \\ \mathscr{C}_{\overline{y}} = \mathscr{C}_1 \cup \cdots \cup \mathscr{C}_s & & X_{\overline{y}} = X_1 \cup \cdots \cup X_r \end{tikzcd}$$

As $\mathscr{C}_U \cong X_U$, it follows that pr_1 and pr_2 are birational (and of course, proper), in particular, surjective. Therefore, $(\mathrm{pr}_1)_{\overline{y}}$ and $(\mathrm{pr}_2)_{\overline{y}}$ are surjective too.

1.22. Lemma. — *In the above setup,*

(i) *For each $1 \leqslant i \leqslant r$, there exists a unique $1 \leqslant j(i) \leqslant t$ so that $T_{j(i)} \to X_i$ is surjective. Also, there exists an open $V \subset X$ meeting X_i densely such that $\mathrm{pr}_2^{-1}(V) \cong V$.*

(*) *Moreoever, $T_{j(i)} \to \mathscr{C}_{\overline{y}}$ is not constant.*

(ii) *For each $1 \leqslant \alpha \leqslant s$, there exists a unique $1 \leqslant \gamma(\alpha) \leqslant t$ so that $T_{\gamma(\alpha)} \to \mathscr{C}_\alpha$ is surjective. Also, there exists an open $W \subset \mathscr{C}$ meeting $\mathscr{C}_\alpha$ densely such that $\mathrm{pr}_1^{-1}(W) \cong W$.*

The three-point assumption P6 (e) is going to be crucially used in the proof of Lemma 1.22 (*). We first note the main corollary–

1.23. Corollary. — $(\mathrm{pr}_1)_{\overline{y}}$ *is quasi-finite, and hence,* pr_1 *is an isomorphism.*

Proof. If $(\mathrm{pr}_1)_{\overline{y}}$ is not quasi-finite, it maps some component T_j to a point in $\mathscr{C}_{\overline{y}}$. By Lemma 1.22 (*), T_j maps to a point under $(\mathrm{pr}_2)_{\overline{y}}$. This is not possible because T_j is a curve sitting in the fibered product $\mathscr{C}_{\overline{y}} \times_{\overline{y}} X_{\overline{y}}$ over the algebraically closed field $\kappa(\overline{y})$. □

1.24. *Proof of Lemma 1.22 (i) and (ii).* By symmetry of the situation, it suffices to prove (i). Item (ii) will follow similarly. Recall that $(\mathrm{pr}_2)_{\overline{y}}$ is surjective. We claim that it is sufficient to find an open $V \subset X$ meeting $X_{\overline{y}}$ densely such that $\mathrm{pr}_2^{-1}(V) \to V$ is an isomorphism. Indeed, if V is such a set, then two distinct components T_j and $T_{j'}$ cannot both map onto X_i under $(\mathrm{pr}_2)_{\overline{y}}$ since $T_j \cap T_{j'}$ is finite, which would imply that the fibers of $\mathrm{pr}_2^{-1}(X_i) \to X_i$ have size at least 2 almost always. This can't be because pr_2 is birational.

By previous constructions, $(X/Y)^{\mathrm{sm}}$ has dense intersection with $X_{\overline{y}}$. Denote

$$A := \{x \in X : \dim \mathrm{pr}_2^{-1}(x) = 0\} \subset X.$$

It is clear that A is open by upper-semicontinuity of fiber dimension. Also A intersects $X_{\overline{y}}$ in a dense set. Indeed, $(\mathrm{pr}_2)_{\overline{y}}$ only has finitely many positive dimensional fibers due to Noetherean reasons. We claim that $V = A \cap (X/Y)^{\mathrm{sm}}$ works. Note that $\mathrm{pr}_2^{-1} V \to V$ is finite birational. The morphism $V \to Y$ is surjective and smooth, hence faithfully flat in particular, therefore V is normal as Y is normal. By Zariski's main theorem, $\mathrm{pr}_2^{-1} V \to V$ is an isomorphism and proof is complete. □

1.25. *Proof of Lemma 1.22 (*).* For each i, we have a unique $T_{j(i)}$ mapping surjectively onto X_i. Assume the contrary that $(\mathrm{pr}_1)_{\overline{y}}(T_{j(i)}) = \{c\}$ for some $c \in \mathscr{C}_{\overline{y}}$. Let $1 \leqslant \alpha < \beta < \gamma \leqslant n$ be such that $x_\alpha = \sigma_\alpha(\overline{y})$, $x_\beta = \sigma_\beta(\overline{y})$, and $x_\gamma = \sigma_\gamma(\overline{y})$ lie in $X_i \cap (X/Y)^{\mathrm{sm}}$. We are using P6 (e) here. If $T \to X' \to X$ is the Stein factorization of pr_2 then $X' \to X$ is a finite modification, hence an isomorphism over the normal locus of X by Zariski's main theorem. In particular, $\mathrm{pr}_2^{-1}(x)$ is connected for any smooth point $x \in (X/Y)^{\mathrm{sm}}$. Note that $t_\alpha = (\tau_\alpha(\overline{y}), \sigma_\alpha(\overline{y})) \in T_{\overline{y}}$ and similarly for β, γ. Finally, put $c_\alpha = (\mathrm{pr}_1)_{\overline{y}}(t_\alpha) = \tau_\alpha(\overline{y})$ and likewise. We remark that $(\mathrm{pr}_2)_{\overline{y}}^{-1}(x) \to (\mathrm{pr}_1)_{\overline{y}}((\mathrm{pr}_2)_{\overline{y}}^{-1}(x))$ is an isomorphism for any $x \in X_{\overline{y}}$ simply because $(\mathrm{pr}_2)_{\overline{y}}^{-1}(x) \subset \mathscr{C}_{\overline{y}} \times_{\overline{y}} \{x\}$.

- *Case 1.* $c \notin \{c_\alpha, c_\beta, c_\gamma\}$. In this case, $z_\alpha = (\mathrm{pr}_1)_{\overline{y}}((\mathrm{pr}_2)_{\overline{y}}^{-1}(x_\alpha))$ is a connected set containing both c_α and c. So, there is a positive dimensional irreducible component of z_α passing through c. A similar statement holds for β and γ, contradicting the fact that $\mathscr{C}_{\overline{y}}$ is a nodal curve. Here, one has to observe that $z_\alpha, z_\beta, z_\gamma$ have no irreducible components in common. This is easy to see because $(\mathrm{pr}_2)_{\overline{y}}^{-1}(x_\alpha), (\mathrm{pr}_2)_{\overline{y}}^{-1}(x_\beta), (\mathrm{pr}_2)_{\overline{y}}^{-1}(x_\gamma)$ are disjoint and $(\mathrm{pr}_1)_{\overline{y}}$, being birational, has only finitely many non-singleton fibers.
- *Case 2.* $c \in \{c_\alpha, c_\beta, c_\gamma\}$. Say $c = c_\alpha$. We see that the curves z_β and z_γ meet at the point $c = c_\alpha \in \mathscr{C}_{\overline{y}}^{\mathrm{sm}}$. This is a contradiction to smoothness. □

We have arranged for pr_1 to be an isomorphism (Corollary 1.23). Therefore, β extends to a morphism $\mathscr{C} \to X$ which is an isomorphism over U.

1.26. Reducing to a family of stable curves. Finally, we replace (X, Z) by $(\mathscr{C}, \beta^{-1}(Z))$. Here, we may lose the finiteness of $Z \to Y$ but that's a non-issue. One can then use induction on dimension to change Y to a regular scheme. The resulting X has very simple singularities, and its desingularization can be carried out by hand. The reader is referred to [dJ, §4.23-4.28] for the explicit blow-ups constructed.

2 Resolution of Embedded Curves

Most of this section is taken from [Stacks, Tag 0BI3] and the results are presented in high generality. The goal is to prove the following theorem–

2.1. Theorem (Embedded resolutions). — *Let X be a Noetherian scheme and $Y \subseteq X$ a reduced closed subscheme of pure dimension 1 with irreducible components $Y_1, Y_2, \ldots, Y_r$. Suppose there are regular integral schemes X_i for each Y_i along with finite morphisms $X_i \to Y_i$. Then there is a finite sequence $X' = X_n \to X_{n-1} \to \cdots \to X_1 \to X$ of blow-ups at closed points such that the proper transform $Y' \subseteq X'$ of Y is a disjoint union of regular integral one dimensional schemes.*

We begin with a few useful lemmas.

2.2. Lemma (Blow-ups of curves are finite). — *Let Y be a one dimensional integral Noetherian scheme and $p \in Y$ a closed point. Then the blow-up $\pi\colon \mathrm{Bl}_p Y \to Y$ is finite.*

There's a quick way to see this. Note that the exceptional divisor, which is an effective Cartier divisor on a curve, is a finite set. Hence, $\mathrm{Bl}_p Y \to Y$ is projective and quasi-finite and so it's finite.

2.3. Lemma (Resolution by blow-ups). — *Let Y be a one dimensional integral Noetherian scheme. Suppose there exists a finite morphism $\pi\colon X \to Y$ for some regular one dimensional (Noetherian) integral scheme X. Then π can be factored as a finite sequence $X \to Y_n \to Y_{n-1} \to \cdots \to Y_1 \to Y_0 = Y$ where Y_i is a blow-up of Y_{i-1} at a closed point for all $1 \leqslant i \leqslant n$ and Y_n is regular.*

Proof. Set $Y_0 = Y$. For each positive integer i, define Y_i to be the blow-up of Y_{i-1} at an arbitrary singular closed point p_{i-1}, if exists, else let p_{i-1} be any (regular) closed point. First we show that π factors as $X \to Y_1 \to Y$.

As X is Noetherian, $\pi^{-1}(p_0) \subseteq X$ is a finite set of closed points, i.e., an effective Cartier divisor. By the universal property of blow-ups (see appendix), it follows that π factors through Y_1. We can iterate the above argument with Y_{i+1} replaced by Y_i for each $i = 0, 1, 2, \ldots$, in that order, to get a (possibly infinite) factorization $X \to \cdots \to Y_2 \to Y_1 \to Y_0 = Y$, where each morphism $Y_{i+1} \to Y_i$ is finite (Lemma 2.2). Of course, strictly speaking, a factorisation being infinite has no meaning. What is meant is an "infinite" commutative diagram of the form

$$\begin{tikzcd}
\cdots \arrow[r, dotted, no head] & Y_2 \arrow[r] & Y_1 \arrow[r] & Y_0 = Y \\
& X \arrow[u] \arrow[ur] \arrow[urr] \arrow[ul, dotted, no head] & &
\end{tikzcd}.$$

Let π_i be the morphism $Y_i \to Y$. Pushing forward all the structure sheaves to Y, we get a chain

$$\mathscr{O}_Y \subseteq \pi_{1,*}\mathscr{O}_{Y_1} \subseteq \pi_{2,*}\mathscr{O}_{Y_2} \subseteq \cdots \subseteq \pi_*\mathscr{O}_X.$$

Indeed, it is appropriate to write $\pi_{i+1,*}\mathscr{O}_{Y_{i+1}} \subseteq \pi_{i,*}\mathscr{O}_{Y_i}$ since $Y_{i+1} \to Y_i$, being a dominant morphism of integral schemes, induces an injection on structure sheaves. This is a sequence of coherent sheaves on a Noetherian scheme, which must stabilize. Consequently, $Y_{i+1} \to Y_i$ is an isomorphism for some i. This means that p_i is already an effective Cartier divisor on Y_i and $\mathscr{O}_{Y_i,p_i}$ is a DVR. Thus, Y_i is regular and the proof is complete. $\square$

2.4. Lemma. — *Let X be a Noetherian scheme. Let $Y \subseteq X$ be an one-dimensional integral closed subscheme such that there is a regular integral scheme Y_1 along with a finite morphism $Y_1 \to Y$. Then there exists a finite sequence $X' = X_n \to X_{n-1} \to \cdots \to X_1 \to X$ of blow-ups at closed points such that the proper transform Y' of Y in X' is a regular integral scheme of dimension 1.*

Proof. Obvious by Lemma 2.3 and Blow-up Closure Lemma (see appendix). $\square$

Let X be a locally Noetherian scheme and $Y, Z \subset X$ be closed subschemes. Assume Y is integral of dimension 1 and Y is not contained in Z. For a closed point $p \in X$, denote the *intersection multiplicity*

$$\operatorname{mult}_p Y \cap Z = \operatorname{length}_{\mathscr{O}_{X,p}} \mathscr{O}_{Y\cap Z,p}.$$

This is finite by the theory of associated primes because $\mathscr{O}_{Y\cap Z,p} = \mathscr{O}_{X,p}/(\mathscr{I}_{Y/X,p} + \mathscr{I}_{Z/X,p})$ has singleton support. We remark that if p is a regular point of Y then $\operatorname{mult}_p Y \cap Z$ is equal to the valuation of $\mathscr{I}_{Z/X,p}\mathscr{O}_{Y,p}$ in the discrete valuation ring $\mathscr{O}_{Y,p}$.

2.5. Lemma. — *Using above notations, let $X' \to X$ be the blow up of X at p and $Y', Z' \subseteq X'$ be the proper transforms of Y, Z. Also assume that $\mathscr{O}_{Y,p}$ is regular. Then*

(1) $Y' \to Y$ is an isomorphism,
(2) Y' intersects the exceptional divisor at exactly one point $q \in X'$,
(3) $\operatorname{mult}_q Y' \cap Z' < \operatorname{mult}_p Y \cap Z$.

Proof. The universal property of blow-ups and the blow-up closure lemma (see appendix) immediately give (1) and (2). Blow-ups can be computed locally, so we may replace X by a small-enough affine open neighborhood $U = \operatorname{Spec} A$ of p such that $Y \cap U$ is cut out by I, $Z \cap U$ is cut out by J, $\mathfrak{m} \subseteq A$ is the maximal ideal corresponding to p, and the image of some section, say $x_1 \in A$, in $\mathscr{O}_{Y,p} = A/I \otimes_A A_\mathfrak{m}$ is a uniformizer. Note that $\mathfrak{m} = I + (x_1)$. Let $x_2, x_3, \ldots, x_n \in A$ be a generating set of I. In this situation, $X' = \operatorname{Bl}_p \operatorname{Spec} A = \operatorname{Proj}_A A[\mathfrak{m}t]$. Consider the affine patch $\operatorname{Spec} A[\mathfrak{m}t/x_1 t]$. The total transform of Y is cut out by $(x_1\frac{x_2t}{x_1t},\dots,x_1\frac{x_nt}{x_1t}) = x_1(\frac{x_2t}{x_1t},\dots,\frac{x_nt}{x_1t})$ in $\operatorname{Spec} A[\mathfrak{m}t/x_1t]$. The factor x_1 corresponds to the exceptional divisor. Hence, the strict transform Y' is cut out by $I' = (\frac{x_2t}{x_1t},\dots,\frac{x_nt}{x_1t})$ in $\operatorname{Spec} A[\mathfrak{m}t/x_1t]$. Therefore, point q corresponds to the maximal ideal $\mathfrak{m}' = (x_1, \frac{x_2t}{x_1t},\dots,\frac{x_nt}{x_1t})$. Now let $f \in J$ be such that its image under $A \to A/I$ has valuation $r = \operatorname{mult}_p Y \cap Z$ at $\mathfrak{m}$. We can write $f = x_1^r u + v$ where $u \notin \mathfrak{m}$ and $g \in (x_2,\dots,x_n)$. Certainly ft/x_1t is an element of the ideal $J' \subseteq A[\mathfrak{m}t/x_1t]$ cutting out Z', as $J \subseteq \mathfrak{m}$. The image of ft/x_1t in $A[\mathfrak{m}t/x_1t]/I'$ is $x_1^k ut/x_1t$. Clearly, ut/x_1t is not a valid element of $A[\mathfrak{m}t/x_1t]$ because $u \notin \mathfrak{m}$. Thus, the valuation of $f \pmod{I'}$ at q in $A[\mathfrak{m}t/x_1t]/I'$ is one less than r. This finishes the proof of (3). ☐

2.6. *Proof of Theorem 2.1.* Applying Lemma 2.4, we can assume Y_i is regular for each $0 \leqslant i \leqslant r$. For every $i \neq j$ and closed point $p \in Y_i \cap Y_j$, consider the number $\operatorname{mult}_p Y_i \cap Y_j$. If the maximum of these numbers is larger than 1, then we can decrease it by blowing up at all the points where the maximum is attained (Lemma 2.5). If the maximum is 1 then we can separate the Y_i's using the same lemma by blowing up at all such points p. Indeed, blow-ups are isomorphism away from the exceptional divisor, so disjoint Y_i's remain disjoint. ☐

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Appendix

Blow-ups. Let $X \hookrightarrow Y$ be a closed embedding corresponding to a finite type quasicoherent sheaf of ideals. The blow-up of $X \hookrightarrow Y$ is a Cartesian diagram, which we call a **blow-up diagram**,

$$\begin{tikzcd} \mathrm{E}_X Y \arrow[r, hook] \arrow[d] & \mathrm{Bl}_X Y \arrow[d] \\ X \arrow[r, hook] & Y \end{tikzcd}$$

such that $\mathrm{E}_X Y$ is an effective Cartier divisor on $\mathrm{Bl}_X Y$, such that any other such Cartesian diagram

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d] & W \arrow[d] \\ X \arrow[r, hook] & Y, \end{tikzcd}$$

where D is an effective Cartier divisor on W, factors uniquely through it:

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d] & W \arrow[d] \\ \mathrm{E}_X Y \arrow[r, hook] \arrow[d] & \mathrm{Bl}_X Y \arrow[d] \\ X \arrow[r, hook] & Y. \end{tikzcd}$$

We call $\mathrm{Bl}_X Y$ the **blow-up** of Y along X and $\mathrm{E}_X Y$ the **exceptional divisor**. If $Z \to Y$ is any morphism then $Z \times_Y \mathrm{Bl}_X Y$ is the **total transform** of Z and $\overline{(Z \setminus X) \times_Y \mathrm{Bl}_X Y}$ the **proper transform** or **strict transform** of Z.

Of course, the above is not really a definition. One needs to construct the blow-up.

$$\mathrm{Bl}_X Y = \mathrm{Proj}_A A \oplus It \oplus I^2 t^2 \oplus \cdots$$
$$\mathrm{E}_X Y = \mathrm{Proj}_A \frac{A}{I} \oplus \frac{I}{I^2} t \oplus \frac{I^2}{I^3} t^2 \oplus \cdots$$

where t is a dummy variable indicating the grading. The **blow-up algebra** or **Rees algebra** $A \oplus It \oplus I^2 t^2 \oplus \cdots$ is also denoted $A[It]$. Further, for any $f \in I$, the zero-degree part of $(A \oplus It \oplus I^2 t^2 \oplus \cdots)_{ft}$ is denoted $A[It/ft]$, which is called an **affine blow-up algebra**. It can be verified that the above construction satisfies the required universal property [FOAG, §23.3.2]. One can then glue these pieces to define blow-up for general schemes using a relative $\mathcal{P}roj$ construction. The blow-up $\mathrm{Bl}_X Y \to Y$ is an isomorphism away from X. Of course, this means that the blow-up is a birational morphism when Y is integral and $X \neq Y$. Another important (but easy) fact is that blow-up preserves irreducibility and reducedness. See [FOAG, §23.2.1, 23.2.A, 23.2.B, 23.2.2, 23.2.C] for details. The most important and useful fact for us about blow-ups is the following–

Blow-up Closure Lemma. *Let $X \hookrightarrow Y$ be a closed embedding of schemes and $Z \to Y$ be any arbitrary morphism. Consider the following commutative diagram*

$$\begin{tikzcd}
 & \mathrm{E}_{\overline{Z}} \arrow[rr, hook, "\text{eff. Cartier}"] \arrow[ddl, dashed] \arrow[dr, hook] & & \overline{Z} \arrow[dr, hook] \arrow[ddl, dashed] & & \\
 & & Z \times_Y \mathrm{E}_X Y \arrow[rr, hook, "\text{loc. prin.}"] \arrow[dll] \arrow[ddr] & & Z \times_Y \mathrm{Bl}_X Y \arrow[dll] \arrow[ddr] & \\
X \times_Y Z \arrow[rr, hook] \arrow[ddr] & & Z \arrow[ddr] & & & \\
 & & & \mathrm{E}_X Y \arrow[rr, hook, "\text{eff. Cartier}"] \arrow[dll] & & \mathrm{Bl}_X Y \arrow[dll] \\
 & X \arrow[rr, hook] & & Y & &
\end{tikzcd}$$

Here, $\overline{Z}$ is the proper transform of Z, i.e., the scheme-theoretic closure of $(Z \times_Y \mathrm{Bl}_X Y) \setminus (Z \times_Y \mathrm{E}_X Y)$ in $Z \times_Y \mathrm{Bl}_X Y$, and $\mathrm{E}_{\overline{Z}}$ is the pullback of $\overline{Z}$ along the closed embedding $Z \times_Y \mathrm{E}_X Y \hookrightarrow Z \times_Y \mathrm{Bl}_X Y$. Then $\overline{Z} = \mathrm{Bl}_{X \times_Y Z} Z$ and $\mathrm{E}_{\overline{Z}}$ is the exceptional divisor on $\overline{Z}$.

Proof. See [Stacks, Tag 080E] for a proof when $Z \to Y$ is a closed embedding. The general case can be proved similarly; see [FOAG, §23.2.6, 23.2.G]. □

2.7. Examples.

(i) *Resolving the planar nodal cubic.* Consider the nodal cubic $y^2 = x^3 + x^2$ in $\mathbb{A}^2$. Then $\mathrm{Bl}_{(0,0)} \mathbb{A}^2$ is cut out by $xY - Xy = 0$ in $\mathbb{A}^2 \times \mathbb{P}^1$ where $\mathbb{A}^2 = \mathrm{Spec}\, k[x, y]$ and $\mathbb{P}^1 = \mathrm{Proj}\, k[X, Y]$. This has two patches– $[X : Y] = [s : 1]$ or $[1 : t]$. The first patch is $U_1 = \mathrm{Spec}\, k[x, y, s]/(sy - x)$ and the second is $U_2 = \mathrm{Spec}\, k[x, y, t]/(y - xt)$. The exceptional divisor is cut out by (y) in U_1 and by (x) in U_2. Therefore, the total transform of $y^2 = x^3 + x^2$ is cut out by $y^2 - s^2 y^2 - s^3 y^3 = 0$ in U_1. We remove the factor of y to get the strict transform– the strict transform of $y^2 = x^3 + x^2$ is cut out by $1 - s^2 - s^3 y = 0$

in U_1. It is easy to see that this is a nonsingular curve. One can similarly compute the local equations on U_2 to verify that the node is resolved.

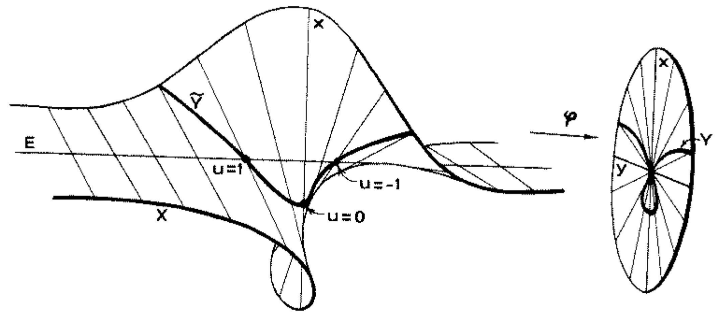

FIGURE 2.1. Resolution of a node through blow-up [Har77].

(ii) *A curve requiring multiple blow-ups.* Consider $yx^2 - y^2 = 0$ in $\mathbb{A}^2$. This is the union of a parabola and the x-axis so it has a multiplicity 2 singularity at the origin. With notations same as the in the previous example, the local equation of the total transform in U_1 is $y(sy)^2 - y^2 = 0$. Hence, the strict transform is cut out by $ys^2 - 1 = 0$ in U_1. This patch is nonsingular. Similarly, the same is cut out by $xt - t^2 = 0$ in $U_2 \cong \operatorname{Spec} k[x,t]$. Of course $(0,0)$ is a singularity of $xt - t^2 = 0$ so we blow-up again. Write $\mathrm{Bl}_{(0,0)} U_2 = \operatorname{Spec} k[x,t,u]/(tu-x) \cup \operatorname{Spec} k[x,t,v]/(t-xv)$ with usual gluing. By similar computations, we get the strict transform of $xt - t^2 = 0$ as $V(u-1) \subset \operatorname{Spec} k[x,t,u]/(tu-x)$ and $V(v-v^2) \subset \operatorname{Spec} k[x,t,v]/(t-xv)$. It is easy to check that these patches are nonsingular. Therefore, two blow-ups were needed to resolve the singularity. We remark that this computation aligns with the statement of Lemma 2.5 (3).

(iii) *Estimating the number of blow-ups required.* Let C be an affine plane curve with a singularity of multiplicity m at the origin and all other points nonsingular. We give an upper bound on the number of blow-ups necessary to get a resolution of singularities. One cannot bound the number of blow-ups required solely in terms of the multiplicity of the singularity. Indeed, consider the following example– choose $n \gg 0$. The blow-up of the hypercusp $y^2 = x^n$ is given by (the Zariski closure of) $u^2 = v^{n-2}$. We see that the multiplicity at the origin stays the same even after $\lfloor \frac{n}{2} \rfloor - 1$ many blow-ups. So, we introduce another parameter– the genus g. Why are we choosing genus and not the degree? the answer is it doesn't really matter because they are both closely related for plane curves. After the first blow-up $\mathrm{Bl}_{(0,0)} C$, the genus drops to $g - m(m-1)/2$. From this point onwards, every blow-up at a singular point decreases the genus by at least 1. Therefore, we are guaranteed to resolve C within $g - m(m-1)/2 + 1$ many blow-ups. Even though this bound is pretty naive, it's sharp– consider $y^2 = x^{2n+1}$, which has a multiplicity 2 singularity at the origin. The blow-up at the origin is given by $y^2 = x^{2n-1}$. So after repeatedly doing the same n times we get a genus 0 conic $y^2 = x$. The genus decreases by exactly $\frac{1}{2}2(2-1) = 1$ on each blow-up, so it follows that $y^2 = x^{2n+1}$ has genus $g = n$. Thus, the number of blow-ups required is *exactly* $n = g - m(m-1)/2 + 1$.

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