On the Divisibility $a! + b! \mid (a + b)!$

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Abstract. In this article, we investigate the pairs of positive integers for which sum of their factorials divides the factorial of their sum and establish a bound on their difference. We also solve the divisibility question over the set of Fibonacci numbers. We conclude by proving that there are infinitely many such pairs of positive integers with difference 2 and conjecture that for any positive integer k there are infinitely many such pairs (a, b) with $|a - b| = k$.

1. INTRODUCTION. Many interesting divisibilities in number theory are of the form $f(a) + f(b) \mid f(a + b)$ where $f : \mathbb{N} \to \mathbb{N}$ is a function. In this article, we let $f(n) = n!$ and so look for pairs (a, b) such that $a! + b! \mid (a + b)!$.

One immediate such pair is (n, n). Call a pair that satisfies the divisibility *good*. To understand the pattern of good pairs, we plot all the good pairs (a, b) with $1 \leq a, b \leq 100$ in the Cartesian plane where the x-axis denotes the a values and the y-axis denotes the b values. With the help of computer we obtain the plot shown in Figure 1.

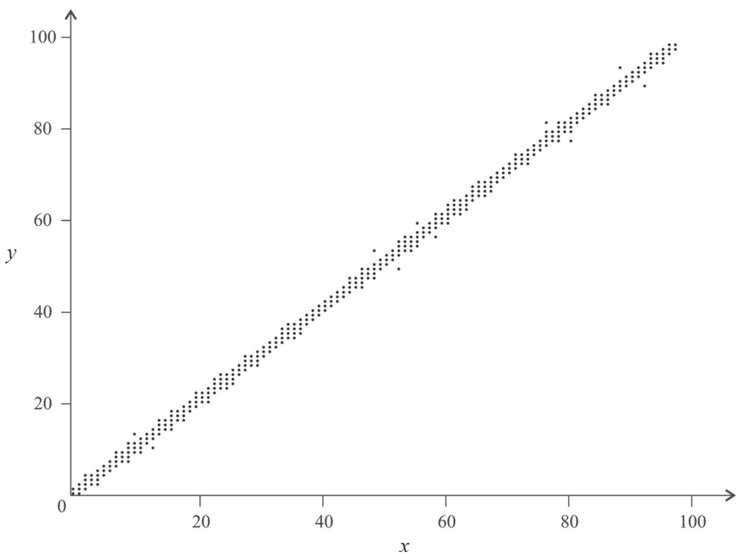

Figure 1. Plot of good pairs.

One immediately notices that the plot resembles the line $x = y$. This motivates us to define the quantity $|a - b|$ as the *deviation* for each good pair (a, b); it can be thought of as a measure of how far a point (a, b) is from the line $x = y$.

For other related results on arithmetic functions and factorials, see, for example, Baczkowski et al. [1], where the authors consider $d(n!)$, $\sigma(n!)$, and $\varphi(n!)$, and [2]

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where the author considers $z(n!)$. Here, $d(n)$ is the number of positive divisors of n, $\sigma(n)$ is the sum of positive divisors of n, $\phi(n)$ is the number of positive integers $k \le n$ that are relatively prime to n, and $z(n)$ is the rank (or the order) of appearance in the Fibonacci sequence.

Definition. If (a, b) is a pair of positive integers that satisfies $a! + b! \mid (a + b)!$, then we say that the pair (a, b) is *good*. For a good pair (a, b) we define the quantity $|a - b|$ as the *deviation* of the pair.

A simple brute force computer checking tells us that the maximum deviation among all pairs (a, b) with $1 \le a, b \le 100$ is 4. This suggests that the deviation of good pairs does not get very large, or at least, that the deviation of a pair (a, b) is very small compared to the sizes of a and b. Hence, it is natural to ask whether we can bound the deviation of a good pair. The following theorem, to be proved in this article, gives an upper bound on the deviations of good pairs. It also explains why the plot resembles the line $x = y$.

Theorem 1. *If a and b are positive integers such that $a \ge b \ge e^{e^{e^{4.22}}}$ and $a! + b!$ divides $(a + b)!$, then*

$$a - b < \frac{b \log\log b}{\log b}.$$

For sufficiently large a and b, using the theorem above, we have that $b \le a \le b + \frac{b \log\log b}{\log b}$. Dividing by b, we obtain

$$1 \le \frac{a}{b} \le 1 + \frac{\log\log b}{\log b}.$$

By taking the limit and using the sandwich theorem, it follows that the ratio a/b converges to 1 as $b \to \infty$. This explains why the plot resembles the $x = y$ line (see Figure 1).

It is evident by looking at the plot that it may be difficult to solve the divisibility in its most general form; often we are interested in solving a problem for a restricted class of integers. Hence, we raise the question: what pairs of Fibonacci numbers are good? The following theorem answers it completely.

Theorem 2. *If a and b are Fibonacci numbers such that $a! + b! \mid (a + b)!$, then*

$$(a, b) \in \{(2, 1), (3, 2), (5, 3), (F_n, F_n)\}$$

up to permutation. Here F_n denotes the nth Fibonacci number.

Once we prove the upper bound on the deviation, a natural question to ask is whether the deviations of good pairs can get arbitrarily large. Or even better, can the deviation be any positive integer? It is trivial that (n, n) and $(n, n + 1)$ are good pairs for all positive integers n; hence, there are infinitely many good pairs with deviation 0 and 1. The following theorem implies that there are infinitely many good pairs with deviation 2.

Theorem 3. *The pair $s = (x^2(x^2 + 2), x^2(x^2 + 2) + 2)$ is good for all positive integers x. In particular, there exist infinitely many good pairs with deviation 2.*

2. BACKGROUND. In what follows, p always denotes a prime. The *first Chebyshev function* is defined as

$$\theta(x) = \sum_{p \le x} \log p.$$

We take the domain of this function to be $\mathbb{R}$. The following estimates for $\theta(x)$ are well known.

Lemma 1 ([5], page 360). $\theta(x) < 1.000081x$ *for all* $x > 0$.

Lemma 2 ([3], page 265). $\theta(x) > 0.985x$ *for all* $x \ge 11927$.

Set $\varepsilon_1 = 0.000081$ and $\varepsilon_2 = 0.015$ from now on. Another very well-known result in the literature, commonly known as Mertens' first theorem, is the following.

Lemma 3 (Mertens' first theorem, [6]). *For all* $x \ge 2$,

$$-1 - \log 4 < \sum_{p \le x} \frac{\log p}{p} - \log x < \log 4.$$

The following lemma is just a weak form of Stirling's approximation.

Lemma 4. *Let* x *be a positive integer. Then*

$$\log x! > x \log x - x.$$

Proof. Observe that $\frac{x^x}{x!}$ is a term in the expansion of e^x:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

Hence, it follows that

$$e^x > \frac{x^x}{x!} \implies x > x \log x - \log x! \implies \log x! > x \log x - x.$$

Our lemma is proved. ■

We will need the following preliminary lemma for proving Theorem 1.

Lemma 5. *If* (a, b) *is a good pair where* $a \ge b \ge 9$*, then its deviation is less than* b.

Proof. Let $k = a - b$. We have that $a! + b! \mid (a + b)!$. Dividing by $b!$ and substituting $a = b + k$, we obtain

$$1 + (b + 1)(b + 2) \cdots (b + k) \mid (b + 1)(b + 2) \cdots (2b + k).$$

Since the left-hand side is relatively prime to $(b + 1)(b + 2) \cdots (b + k)$, we can divide the right-hand side by $(b + 1)(b + 2) \cdots (b + k)$ to get

$$1 + (b + 1)(b + 2) \cdots (b + k) \mid (b + k + 1)(b + k + 2) \cdots (2b + k).$$

The right-hand side is the product of b consecutive integers and the left-hand side is relatively prime to $k!$, since the product of k consecutive integers is divisible by $k!$. Assume to the contrary that $k \geq b$. Then we must have that

$$1 + (b+1)(b+2)\cdots(b+k) \mid \frac{(b+k+1)(b+k+2)\cdots(2b+k)}{b!}.$$

Thus,

$$(b+1)(b+2)\cdots(b+k) < \frac{(b+k+1)(b+k+2)\cdots(2b+k)}{b!}.$$

The above implies that

$$b^k < \frac{(2b+k)^b}{b!}.$$

Taking logarithms and using Lemma 4, we get

$$\begin{aligned} k \log b &< b\log(2b+k) - (b\log b - b) \\ &= b\log(2 + \tfrac{k}{b}) + b. \end{aligned}$$

Let $x = \frac{k}{b}$; we know that $x \geq 1$ by assumption. Dividing the above by b, we obtain

$$f(x) = x\log b - \log(2+x) < 1.$$

The left-hand side is increasing in x for all $b \geq 9$ since $f'(x) = \log b - \frac{1}{2+x}$. We know that $x \geq 1$. Hence, $\log b - \log 3 = f(1) \leq f(x) < 1$, which is clearly false since $b \geq 9$. So our assumption was wrong and the lemma is proved. ∎

3. BOUND ON DEVIATION. In this section, we prove Theorem 1.

Set-up. Set $k = a - b$. By Lemma 5, we know that $k < b$. Assume to the contrary that there are infinitely many good pairs (a, b) such that $k \geq \frac{b \log\log b}{\log b}$. In what follows we always assume $k \geq \frac{b\log\log b}{\log b}$. We have that $a! + b! \mid (a+b)!$. Dividing by $b!$ and substituting $a = b + k$, we obtain

$$1 + (b+1)(b+2)\cdots(b+k) \mid (b+1)(b+2)\cdots(2b+k).$$

Since the left-hand side is relatively prime to $(b+1)(b+2)\cdots(b+k)$, we can divide the right-hand side by $(b+1)(b+2)\cdots(b+k)$ to get

$$1 + (b+1)(b+2)\cdots(b+k) \mid (b+k+1)(b+k+2)\cdots(2b+k).$$

Let $N{=}1 + (b+1)(b+2)\cdots(b+k)$ and $\prod_{i=1}^{n} p_i^{\alpha_i}$ be the prime factorization of N, where $p_1 < p_2 < \cdots < p_n$ are primes and the α_i are positive integers.

Obtaining the main bound. Clearly $N - 1$ is the product of k consecutive positive integers. Therefore all the prime factors of N must be greater than k. We have that

$$\prod_{i=1}^{n} p_i^{\alpha_i} \mid (b+k+1)(b+k+2)\cdots(2b+k).$$

Note that if $p_i^2 \mid b+j$ for some $k+1 \le j \le b+k$ and $1 \le i \le n$, then $k^2 < p_i^2 \le b+j \le 2b+k$, which implies $k^2 - k \le 2b$ as $p_i > k$, which cannot be true as $b > e^{e^{e^{4.22}}}$ and $k \ge \frac{b \log\log b}{\log b}$. Therefore, α_i can be at most the number of multiples of p_i in $\{b+k+1, b+k+2, \ldots, 2b+k\}$. Hence we have that

$$\alpha_i \le \left\lfloor \frac{2b+k}{p_i} \right\rfloor - \left\lfloor \frac{b+k}{p_i} \right\rfloor < \frac{2b+k}{p_i} - \left(\frac{b+k}{p_i} - 1 \right) = \frac{b}{p_i} + 1,$$

and, in particular, that $\alpha_i = 1$ if $p_i \ge b$. Thus, we have that

$$b^k < \prod_{i=1}^{k} (b+i) < N = \prod_{i=1}^{n} p_i^{\alpha_i} \le \prod_{p \in (k,b)} p^{\frac{b}{p}+1} \cdot \prod_{p \in \{b\}} p \cdot \prod_{b+k < p \le 2b+k} p.$$

Taking logarithms, we get

$$k \log b < \theta(2b+k) - \theta(b+k) + \theta(b) - \theta(k) + b \left(\sum_{k<p<b} \frac{\log p}{p} \right). \tag{1}$$

Estimating both sides of (1). We estimate the right-hand side of (1) term by term. It is easy to check that we have $k \ge 11927$ under the assumptions $k \ge \frac{b \log\log b}{\log b}$ and $b \ge e^{e^{e^{4.22}}}$. By Lemmas 1 and 2, we have

$$\begin{aligned}
\theta(2b+k) - \theta(b+k) + \theta(b) - \theta(k) &\le (1+\varepsilon_1)(2b+k) - (1-\varepsilon_2)(b+k) \\
&\quad + (1+\varepsilon_1)b - (1-\varepsilon_2)k \\
&= b(2+3\varepsilon_1+\varepsilon_2) - k(1-2\varepsilon_2-\varepsilon_1) \\
&< b(2+3\varepsilon_1+\varepsilon_2).
\end{aligned}$$

Using Lemma 3, we have

$$\sum_{k<p<b} \frac{\log p}{p} < \log b - \log k + 1 + 2\log 4 = \log\left(\frac{b}{k}\right) + 1 + 2\log 4.$$

Summing up. By combining the estimates, it follows that

$$k \log b < b(2+3\varepsilon_1+\varepsilon_2) + b\left(\log\left(\frac{b}{k}\right) + 1 + 2\log 4 \right).$$

Simplifying, we get

$$k \log b < b \log\left(\frac{b}{k}\right) + 4.22b.$$

Dividing by b, we obtain

$$\frac{k \log b}{b} < \log\left(\frac{b}{k}\right) + 4.22.$$

The left-hand side is increasing in k and the right-hand side is decreasing in k. Using the assumption that $k \geq \frac{b \log\log b}{\log b}$, we have that

$$\log\log b < \log\left(\frac{\log b}{\log\log b}\right) + 4.22,$$

which is evidently false for all $b \geq e^{e^{e^{4.22}}}$. This gives a contradiction and so the desired result is proved.

4. GOOD PAIRS OF FIBONACCI NUMBERS. The Fibonacci sequence is defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$. We are interested in solving the divisibility $a! + b! \mid (a+b)!$ over the Fibonacci numbers. It is well known that

$$F_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\overline{\varphi} = -\varphi^{-1} = \frac{1-\sqrt{5}}{2}$; this is known as *Binet's formula*.

Obviously $(a, b) = (F_m, F_m)$ is a solution for all $m \geq 1$. Without loss of generality, assume that $a > b \geq 5$; we will deal with the cases $b = 1, 2, 3$ at the end. Note that $b \neq 4$ since 4 is not a Fibonacci number. Let $b = F_n$ and $a - b = k$. So we have that $k = a - b \geq F_{n+1} - F_n = F_{n-1}$. For now, assume that $n \geq 23$. Hence, $b = F_n \geq F_{23} = 28657$ and $k \geq F_{n-1} \geq F_{22} = 17711$. Inequality (1) gives

$$k \log b < \theta(2b+k) - \theta(b+k) + \theta(b) - \theta(k) + b\left(\sum_{k<p<b} \frac{\log p}{p}\right).$$

Using Lemmas 1–3, we get

$$\begin{aligned} k \log b &< \theta(2b+k) - \theta(b+k) + \theta(b) - \theta(k) + b\left(\sum_{k<p<b} \frac{\log p}{p}\right) \\ &< (1+\varepsilon_1)(3b+k) - (1-\varepsilon_2)(b+2k) + b(\log \tfrac{b}{k} + 4) \\ &< 2.1b - 0.9k + b(\log \tfrac{b}{k} + 4) \\ &< 2.1b - 0.9k + b(1+4) \\ &= 7.1b - 0.9k. \end{aligned} \tag{2}$$

Note that

$$\frac{F_n}{F_{n-1}} = \frac{\varphi^n - (-\varphi^{-1})^n}{\varphi^{n-1} - (-\varphi^{-1})^{n-1}} = \varphi + (-1)^{n+1}\frac{\varphi^{-n} + \varphi^{-n+2}}{\varphi^{n-1} + (-1)^n\varphi^{-n+1}}.$$

From the above identity, we can see that $\frac{F_n}{F_{n-1}}$ is greater than φ if and only if n is odd. Hence,

$$\frac{b}{k} \leq \frac{F_n}{F_{n-1}} \leq \frac{F_9}{F_8} = \frac{34}{21} < 1.62$$

since $n \geq 23$. Dividing (2) by k, we get

$$\log b < 7.1 \cdot \tfrac{b}{k} - 0.9 < 7.1 \cdot 1.62 - 0.9 = 10.602$$

which implies $b < e^{10.602} < 40216$. We have now proved that if $b \geq 28657$ then $b \leq 40215$, so $b \leq 40215$ in any case. Since $F_{23} = 28657$ and $F_{24} = 46368$, we get that $b \leq F_{23}$ and $n \leq 23$. We next bound k. Again we make use of inequality (1):

$$\begin{aligned}
k \log b &< \theta(2b+k) - \theta(b+k) + \theta(b) - \theta(k) + b\left(\sum_{k<p<b} \frac{\log p}{p}\right) \\
&< \theta(2b+k) + \theta(b) + b\left(\sum_{k<p<b} \frac{\log p}{p}\right) \\
&< (1+\varepsilon_1)(3b+k) + b(\log \tfrac{b}{k} + 4) \\
&< (1+\varepsilon_1)(3b+k) + 5b \\
&< 8.1b + (1+\varepsilon_1)k.
\end{aligned}$$

We know that $b \leq F_{23}$ and, using our assumption that $b \geq 5$, we obtain

$$k \log 5 \leq k \log b < 8.1b + (1+\varepsilon_1)k \leq 8.1 \cdot F_{23} + (1+\varepsilon_1)k.$$

Hence,

$$k < \frac{8.1 \cdot F_{23}}{\log 5 - 1 - \varepsilon_1} < 3.9 \times 10^5.$$

This shows that $a = b + k < 28657 + 3.9 \times 10^5 = 418657$. Now we have that $(a, b) \in \{1, 2, \ldots, 418657\} \times \{1, 2, \ldots, 28657\}$. By an easy computer check, we conclude that the solutions are $(a, b) \in \{(2, 1), (3, 2), (5, 3)\}$. What remains is to check the cases $b = 1, 2, 3$. For $b = 1$, we obtain that $a! + 1 \mid \frac{(a+1)!}{a!} = a + 1$, which implies $a = 1, 2$. If $b = 2$, again, $a! + 2 \mid \frac{(a+2)!}{a!/2} = 2(a+1)(a+2)$, which forces $a \leq 4$ due to size reasons, i.e., $a! + 2 \leq 2(a+1)(a+2)$ is false for all $a \geq 5$. If $b = 3$, we obtain that $a! + 6 \mid \frac{(a+3)!}{a!/6} = 6(a+1)(a+2)(a+3)$, which again by the same reasoning implies that $a \leq 6$. It can be verified that all the valid solutions are already found and that they indeed work.

5. ADMISSIBLE VALUES OF DEVIATIONS. We have seen that the deviation of a good pair is small compared to its components and established a bound on the deviation. In this section, we investigate the natural question of whether the deviations of good pairs can take on any value. That is, for any positive integer k, does there exist a good pair with deviation k? We will prove that the deviation can be 0, 1, and 2 for infinitely many good pairs. It is clear that (n, n) is a good pair with deviation 0. Also it is not hard to verify that $(n, n+1)$ is good for all n: note that $n! + (n+1)! \mid (2n+1)!$ is simply $n + 2 \mid (n+1)(n+2)\cdots(2n+1)$ after dividing both sides by $n!$. Table 1 gives examples of good pairs with different deviations.

We are going to use the idea in [**4**, Lemma 10] to prove Theorem 3, which states that there are infinitely many good pairs with deviation 2.

Table 1. Examples of good pairs with different deviations.

Deviation	Good pairs
2	(3, 5), (8, 10), (10, 12), (15, 17), (17, 19), (21, 23)
3	(11, 14), (57, 60), (112, 115), (133, 136), (205, 208)
4	(50, 54), (78, 82), (90, 94), (126, 130), (137, 141), (148, 152)

Proof. Let $a = x^2(x^2 + 2)$. We want to prove that $a! + (a + 2)! \mid (2a + 2)!$ for all positive integers x. It is easily verified that this holds for $x \in \{1, 2, \dots, 9\}$. So let us assume $x \geq 10$. Dividing by $a!$, we get $1 + (a + 1)(a + 2) \mid (a + 1)(a + 2)\cdots(2a + 2)$. The left-hand side $1 + (a + 1)(a + 2)$ factors as $(x^2 - x + 1)(x^2 + x + 1)(x^4 + 3x^2 + 3)$. The right-hand side is

$$\begin{aligned} B &= (a + 1)(a + 2)\cdots(2a + 2) \\ &= (x^4 + 2x^2 + 1)(x^4 + 2x^2 + 2)\cdots(2x^4 + 4x^2 + 2). \end{aligned}$$

Notice that $x^4 + 3x^2 + 3$ is a term in the product above. We claim that $\gcd(x^4 + 3x^2 + 3, (x^2 - x + 1)(x^2 + x + 1)) = 1$ for all integers x. To prove this, it suffices to check that $x^4 + 3x^2 + 3$ and $x^2 + x + 1$ are relatively prime for all integers x since $x^2 - x + 1$ is obtained from $x^2 + x + 1$ by changing x to $-x$, an operation which does not change $x^4 + 3x^2 + 3$. We have

$$(x + 1)(x^4 + 3x^2 + 3) - (x^3 + 2x + 1)(x^2 + x + 1) = 2.$$

Thus any prime dividing both $x^2 + x + 1$ and $x^4 + 3x^2 + 3$ must be 2, which is false since $x^2 + x + 1$ is odd.

Therefore $x^4 + 3x^2 + 3$ and $(x^2 - x + 1)(x^2 + x + 1)$ are relatively prime. As $x^4 + 3x^2 + 3$ is a factor of the product $(a + 1)(a + 2)\cdots(2a + 2)$, it follows that $x^4 + 3x^2 + 3$ divides B; hence, it suffices to prove that $A = (x^2 - x + 1)(x^2 + x + 1)$ divides B for all x. Let p be a prime divisor of A. Clearly $p \leq x^2 + x + 1$. Note that there are

$$\left\lfloor \frac{2a + 2}{p} \right\rfloor - \left\lfloor \frac{a}{p} \right\rfloor \geq \frac{a + 2}{p} - 2 \geq \frac{x^2(x^2 + 2) + 2}{x^2 + x + 1} - 2 > x^2 - x - 1$$

multiples of p in the product $B = (a + 1)(a + 2)\cdots(2a + 2)$. Therefore the exponent of p in the prime factorization of B is at least $x^2 - x - 1$. Now, the exponent of p in the prime factorization of A is at most

$$\log_p A \leq \log_2(x^4 + x^2 + 1) < \log_2(2x^4) < 8\log x + 1 < 8x + 1,$$

which is less than $x^2 - x - 1$ for all $x \geq 10$. Hence, our theorem is proved. ∎

We end the article with an open problem for readers:

Conjecture 1. *For any positive integer k, there exist infinitely many good pairs with deviation k.*

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