FINITE GENERATION IN COHOMOLOGY OF ARTIN STACKS

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Abstract. The cohomology algebra of an Artin stack need not be finitely generated. We prove a finiteness theorem for the self-extensions of its intersection complex instead. Let X be a quasi-separated Artin stack of finite type over an algebraically closed field k, and let ℓ be invertible in k. Assume that X has affine stabilizers and that either $\operatorname{char} k = 0$ or X has quasi-affine diagonal. Then the algebra

$$E_X := \operatorname{Ext}^*_X(\mathrm{IC}_X, \mathrm{IC}_X)$$

is finitely generated over $\mathbf{Q}_\ell$ and is left and right noetherian, where IC_X is the intersection complex. When X is $\mathbf{Q}_\ell$-rationally smooth, $E_X = \mathrm{H}^*(X, \mathbf{Q}_\ell)$, so this gives finite generation of its cohomology algebra. The proof uses Chern classes of vector bundles on projective modifications. We also compute the cohomology of torus gerbes over affine nodal curves, obtaining examples where the reduced cohomology algebra is not finitely generated. The Legendre elliptic gerbe shows that finite generation fails without the affine-stabilizer hypothesis.

1. Introduction

For an Artin stack X of finite type over an algebraically closed field k and a prime ℓ invertible in k, the groups $\mathrm{H}^i(X, \mathbf{Q}_\ell)$ need not vanish in large degrees. This occurs already for classifying stacks. For quotient stacks by affine algebraic groups, equivariant cohomology supplies a finitely generated algebra of characteristic classes over which cohomology is finite; see [IZ16, §4]. Hansen has asked whether the cohomology algebra of a finite-type Artin stack with affine diagonal is always finitely generated [Han].

The cohomology algebra need not be finitely generated, even modulo nilpotents. We prove that the algebra of self-extensions of the intersection complex is finitely generated and left and right noetherian under the hypotheses below. When X is $\mathbf{Q}_\ell$-rationally smooth, the intersection complex is a shifted constant sheaf on each connected component, and this gives finite generation of $\mathrm{H}^*(X, \mathbf{Q}_\ell)$.

Let IC_X denote the intersection complex with $\mathbf{Q}_\ell$-coefficients, using perverse normalization on each irreducible component of X_{red}; its construction is recalled in Section 2. Ext groups are taken in $D^b_c(X, \mathbf{Q}_\ell)$, with Yoneda multiplication.

Theorem A (Intersection complexes have noetherian extension algebras)**.** *Let k be an algebraically closed field, let ℓ be invertible in k, and let X be a quasi-separated Artin stack of finite type over k with affine stabilizers. Assume that* $\operatorname{char} k = 0$ *or that X has quasi-affine diagonal. Then*

$$E_X := \operatorname{Ext}^*_X(\mathrm{IC}_X, \mathrm{IC}_X)$$

is a finitely generated $\mathbf{Q}_\ell$-algebra and is left and right noetherian. For every $K \in D^b_c(X, \mathbf{Q}_\ell)$, the graded module $\operatorname{Ext}^*_X(\mathrm{IC}_X, K)$ *is finite on the right over E_X, and*

$$\operatorname{Ext}^*_X(K, \mathrm{IC}_X), \qquad \mathrm{H}^*(X, \mathrm{IC}_X \otimes^{\mathrm{L}} K)$$

are finite on the left over E_X.

The algebra E_X need not be graded-commutative. It acts on intersection cohomology by endomorphisms of IC_X, rather than by a cup product on intersection cohomology itself. Restriction also makes E_U finite on both sides over E_X for every open $U \subset X$ (Corollary 4.7).

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An Artin stack is $\mathbf{Q}_\ell$-rationally smooth if, smooth-locally on a presentation, its intersection complex is the shifted constant sheaf. On a reduced pure-dimensional component of dimension d, this means $\mathrm{IC}_X \simeq \mathbf{Q}_{\ell,X}[d]$. Equivalently, the fundamental class identifies its dualizing complex with $\mathbf{Q}_{\ell,X}(d)[2d]$. In this case,

$$E_X = \mathrm{Ext}^*_X(\mathbf{Q}_{\ell,X}[d], \mathbf{Q}_{\ell,X}[d]) = \mathrm{H}^*(X, \mathbf{Q}_\ell), \qquad \mathrm{IC}_X \otimes^{\mathrm{L}} K[-d] \simeq K.$$

Thus Theorem A gives the following corollary.

Corollary B. *Let k be an algebraically closed field, let ℓ be invertible in k, and let X be a quasi-separated Artin stack of finite type over k with affine stabilizers. Assume that $\mathrm{char}\, k = 0$ or that X has quasi-affine diagonal. If X is $\mathbf{Q}_\ell$-rationally smooth, then the graded algebra*

$$\mathrm{H}^*(X, \mathbf{Q}_\ell)$$

is finitely generated over $\mathbf{Q}_\ell$. Moreover, for every $K \in D^b_c(X, \mathbf{Q}_\ell)$, the graded module $\mathrm{H}^*(X, K)$ is finite over $\mathrm{H}^*(X, \mathbf{Q}_\ell)$.

Every smooth Artin stack is $\mathbf{Q}_\ell$-rationally smooth, so the conclusion holds for every smooth X satisfying the stated hypotheses. In characteristic zero, rational smoothness does not depend on ℓ (Remark 2.6).

We use the following geometric result:

Theorem C. *Let k be an algebraically closed field, and let X be a quasi-separated reduced irreducible Artin stack of finite type over k with affine stabilizers. Assume that $\mathrm{char}\, k = 0$ or that X has quasi-affine diagonal. There exist a representable projective birational morphism*

$$f : Y \longrightarrow X$$

with Y reduced and irreducible, and a vector bundle V on Y such that, for every geometric point $y \to Y$, the kernel of $G_y \to \mathrm{GL}(V_y)$ contains no positive-dimensional torus. If X has quasi-affine diagonal, V may be chosen faithful; in particular, $Y \simeq [P/\mathrm{GL}_N]$ for a finite-type algebraic space P. In characteristic zero, Y may additionally be chosen smooth.

For quasi-affine diagonal, an affine smooth presentation allows us to construct a faithful vector bundle after a projective modification (see Definition 3.2). This construction works in every characteristic.

In characteristic zero, assuming only that the stabilizers are affine, we begin with a faithful vector bundle on a dense open and extend it after modification. The extension may lose faithfulness along the boundary. After resolving the boundary, we add the line bundles of its divisors to ensure that no positive-dimensional stabilizer torus acts trivially on the resulting bundle. In both cases, the Chern classes of the bundle generate an algebra over which cohomology with bounded constructible coefficients is finite.

Noetherianity of geometric extension algebras in equivariant settings is established in [Kat17, Lemma 1.3]. We require neither a global quotient presentation nor a vector bundle as in Theorem C on X. Characteristic classes make the extension algebra of $\mathrm{R} f_* \mathrm{IC}_Y$ finite on both sides over a polynomial algebra, and IC_X is a summand of this pushforward. Noetherianity passes to its idempotent corner. Since f is projective, this summand exists over every algebraically closed field, without a separated-diagonal hypothesis; the proof uses decomposition and relative hard Lefschetz on scheme presentations [Sun12, Sun17, SZ16]. Using IC_Y avoids resolution in positive characteristic. For a derived version, see Remark 4.8.

Without rational smoothness, finite generation of cohomology can fail. For a graded-commutative rational algebra, we write $\sqrt{(0)}$ for its nilradical.

Theorem D. *Let k be algebraically closed, let ℓ be invertible in k, and let L be a free abelian group of rank $r \geq 1$. Choose $m \geq 1$ automorphisms $\sigma_1, \ldots, \sigma_m \in \mathrm{GL}(L)$. Form an affine integral nodal curve C by identifying m disjoint pairs of distinct points of $\mathbf{A}^1_k$. Let $T \to C$ be the rank-r torus obtained from* *the split torus on the normalization by gluing the fibers over the ith pair via the automorphism whose pullback on characters is σ_i.*

Put $X = B_C T$, $\Gamma = \langle \sigma_1, \dots, \sigma_m \rangle$, and

$$P = \mathrm{Sym}_{\mathbf{Q}_\ell}(L \otimes \mathbf{Q}_\ell), \qquad d : P \longrightarrow P^{\oplus m}, \quad p \longmapsto \big((\sigma_i - 1)p\big)_{i=1}^{m}.$$

With $A = P^\Gamma$ and $M = \operatorname{coker} d$, there is an isomorphism of graded algebras

$$\mathrm{H}^*(X, \mathbf{Q}_\ell) \simeq A \oplus M, \qquad M^2 = 0,$$

where A_n has cohomological degree $2n$, M_n has degree $2n+1$, and the A-action on M is induced by multiplication on $P^{\oplus m}$. The nilradical is M, and each cohomology group is finite-dimensional. If $m \geq 2$, this cohomology algebra is finitely generated if and only if Γ is finite.

The automorphisms in Theorem D need not commute. The next example shows that failure of finite generation need not be confined to the nilradical.

Corollary E. *Let k be an algebraically closed field of arbitrary characteristic. There exist an affine integral curve C/k, whose normalization is $\mathbf{A}^1_k$ and whose only singularities are four ordinary nodes, and a rank-sixteen torus $T \to C$ such that $X = B_C T$ is of finite type with affine diagonal, and, for every prime $\ell \neq \operatorname{char} k$, the algebra*

$$\mathrm{H}^*(X, \mathbf{Q}_\ell)/\sqrt{(0)}$$

is not finitely generated. Its nilradical is its odd cohomology and has square zero. Each individual cohomology group is finite-dimensional. Nevertheless, $\mathrm{H}^*(X, \mathbf{Z}/\ell^a)$ is finitely generated for every $a \geq 1$.

The torus in Corollary E is specified by four automorphisms of its character lattice. Normalization identifies the even cohomology with their polynomial invariant ring. We choose the automorphisms from Totaro's explicit sixteen-dimensional representation of $\mathbf{G}_\mathrm{a}^4$ with non-finitely generated invariants [Tot08, Corollary 7.1].

Section 2 establishes the cohomological criteria, and Section 3 proves the geometric Theorem C. Section 4 proves Theorem A and Corollary 4.7 over arbitrary algebraically closed fields. Section 5 deduces Corollary B, gives an alternative proof using traces, and explains where characteristic zero is used. Section 6 proves Theorem D and Corollary E. Finally, Section 7 exhibits a smooth stack with proper diagonal whose cohomology is not finitely generated, showing that the affine-stabilizer hypothesis cannot be discarded.

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2. Chern classes and cohomological retractions

Setup 2.1 (Standing hypotheses)**.** Throughout, k is algebraically closed and ℓ is invertible in k. Unless a stronger hypothesis is stated, stacks are quasi-separated and of finite type over k, and morphisms are of finite type. Quasi-separated means that both the diagonal and its diagonal are quasi-compact; no separatedness assumption on the diagonal is imposed. We write G_x for the group scheme of automorphisms of a geometric point x.

Convention 2.2 (Cohomology and Tate twists)**.** Cohomology means ℓ-adic étale cohomology, computed on the lisse–étale site, with its cup product. We write $D^b_c(X, \mathbf{Q}_\ell)$ for the homotopy category of bounded constructible Cartesian-adic complexes in the enhanced formalism of Liu–Zheng [LZ24, Definition 7.1.1 and §7.4], with rational coefficients obtained by inverting ℓ in the $\mathbf{Z}_\ell$-adic category. Ext groups are computed in this category, rather than in the bounded derived category of the abelian category of constructible Cartesian sheaves. Bounded constructibility can be checked on a finite smooth presentation; see [LZ24, Lemma 6.4.2, Definition 6.4.3 and §7.4].

We use the six operations of [LZ24, §7.2]. Constructibility and duality are treated in [LZ24, Propositions 7.4.1–7.4.2, 7.5.2 and 7.5.4]; smooth descent and smooth base change are [LZ24, Propositions 7.2.2(P4) and 7.2.3]. The perverse t-structure is smooth-local by [LZ24, §8.3, especially Lemma 8.3.1]. We suppress Tate twists in polynomial descriptions of cohomology rings after a choice of trivialization of $\mathbf{Q}_\ell(1)$ over k, but retain them in duality formulas.

Definition 2.3 (Intersection complexes)**.** For a reduced irreducible stack X of dimension d, let $j : U \hookrightarrow X$ be its smooth locus. We use the perverse normalization

$$\mathrm{IC}_X := j_{!*}\mathbf{Q}_{\ell,U}[d],$$

where $j_{!*}$ is intermediate extension in the category of perverse $\mathbf{Q}_\ell$-sheaves. For arbitrary X, let $i_a : X_a \hookrightarrow X$ be its reduced irreducible components and define

$$\mathrm{IC}_X := \bigoplus_a (i_a)_* \mathrm{IC}_{X_a},$$

with each IC_{X_a} normalized by $\dim X_a$. For the reduction $i : X_{\mathrm{red}} \hookrightarrow X$ we have $\mathrm{IC}_X = i_* \mathrm{IC}_{X_{\mathrm{red}}}$. This construction is compatible with smooth presentations: if $p : V \to X$ is smooth of pure relative dimension r, then $p^* \mathrm{IC}_X[r] \simeq \mathrm{IC}_V$; see [LO09, §4 and Lemma 6.2].

Example 2.4. If X is smooth of pure dimension d, then

$$\mathrm{IC}_X = \mathbf{Q}_{\ell,X}[d].$$

Thus every smooth Artin stack is $\mathbf{Q}_\ell$-rationally smooth.

Remark 2.5 (Rational smoothness)**.** For a stack X, put $\omega_X = a_X^! \mathbf{Q}_\ell$, where $a_X : X \to \operatorname{Spec} k$. We call X *$\mathbf{Q}_\ell$-rationally smooth* if each connected component has a pure dimension d and, smooth-locally on a presentation, its intersection complex is the constant sheaf shifted by d, with the usual additional shift by the relative dimension of the presentation. Rational smoothness is insensitive to nilpotent thickenings. We show in Remark 5.1 that every reduced connected rationally smooth stack is irreducible. On a reduced pure-dimensional component of dimension d, it is equivalent to the isomorphism

$$\mathbf{Q}_{\ell,X}(d)[2d] \xrightarrow{\sim} \omega_X \tag{2.1}$$

induced by the fundamental class. The equivalence with $\mathrm{IC}_X \simeq \mathbf{Q}_{\ell,X}[d]$ is the usual rational-smoothness criterion; see [HaiR20, Proposition 2.1]. These statements extend from schemes by smooth descent. The orientation in (2.1) restricts to the standard orientation on the smooth locus.

Remark 2.6 (Independence of ℓ in characteristic zero)**.** If $\operatorname{char} k = 0$, then $\mathbf{Q}_\ell$-rational smoothness is independent of the prime ℓ. Indeed, smooth locality reduces the assertion to affine schemes of finite type over k. Descend such a scheme to a finitely generated subfield $k_0 \subset k$ and choose an embedding $k_0 \hookrightarrow \mathbf{C}$. Invariance under extension of algebraically closed fields reduces the question to the resulting complex scheme; see [SGA4, Exposé XVI, Corollaire 1.6]. By étale–classical comparison [SGA4, Exposé XVI, Théorème 4.1], rational smoothness there is equivalent to its analytification being a rational homology manifold. This condition is independent of ℓ: the local cohomology groups with $\mathbf{Q}_\ell$-coefficients are obtained from those with $\mathbf{Q}$-coefficients by the faithfully flat extension $\mathbf{Q} \subset \mathbf{Q}_\ell$.

Definition 2.7 (Safe morphisms)**.** Following Boyarchenko–Drinfeld [BD14, Definition B.31], a morphism $g : Z \to W$ is *safe* if, at every geometric point $z \to Z$, the reduced identity component of $\ker(G_z \to G_{g(z)})$ is unipotent.

We require the following form of the boundedness theorem for safe stacks:

Lemma 2.8 (Bounded cohomology for safe stacks)**.** *Let Z be a quasi-separated Artin stack of finite type over k, with affine stabilizers whose reduced identity components are unipotent. For every $K \in D^b_c(Z, \mathbf{Q}_\ell)$, the complex $\mathrm{R\Gamma}(Z, K)$ is bounded and has finite-dimensional cohomology.*

Proof. By [HR15, Proposition 2.6(1)], there is a finite stratification of Z by locally closed substacks admitting representable quasi-affine morphisms to classifying stacks $B\mathrm{GL}_n$. Every stratum has affine diagonal. On each stratum the assertion follows from [BD14, Definition B.31 and Lemma B.33].

Over an algebraically closed field, the reduction G_{red} of a finite-type group scheme G is a smooth subgroup. The map $BG_{\mathrm{red}} \to BG$ is representable and a finite universal homeomorphism, with geometric fiber G/G_{red}, so topological invariance removes the nonreduced structure from the étale calculation. The connected unipotent group $(G_{\mathrm{red}})^0$ has a cohomologically trivial classifying stack. The remaining finite component group has exact invariants over $\mathbf{Q}_\ell$: averaging divides by its order, which is invertible in $\mathbf{Q}_\ell$ even when divisible by ℓ. Thus the argument applies with rational coefficients without any tameness or prime-to-ℓ restriction on the component groups.

We pass from the strata to Z by induction. For an open stratum $j : U \hookrightarrow Z$ with closed complement $i : Z_0 \hookrightarrow Z$, localization gives

$$\mathrm{R\Gamma}(Z_0, i^! K) \longrightarrow \mathrm{R\Gamma}(Z, K) \longrightarrow \mathrm{R\Gamma}(U, j^* K).$$

The complex $i^! K$ is bounded constructible: this may be checked on a finite smooth presentation, where i becomes a closed embedding of finite-type schemes. The induction hypothesis applies to the first term and the stratum case to the last. □

Definition 2.9 (Detecting stabilizer tori)**.** A vector bundle E on Z *detects stabilizer tori* if, for every geometric point z, the kernel of

$$G_z \longrightarrow \mathrm{GL}(E_z)$$

contains no positive-dimensional torus. For affine stabilizers this is equivalent to its reduced identity component being unipotent. Thus, when Z has affine stabilizers and E has rank N, E detects stabilizer tori if and only if its classifying morphism $g : Z \to B\mathrm{GL}_N$ is safe.

Proposition 2.10 (Finiteness over the algebra generated by Chern classes)**.** *Let Z have affine stabilizers, and let E be a rank-N vector bundle on Z detecting stabilizer tori. For every $K \in D^b_c(Z, \mathbf{Q}_\ell)$, the graded module $\mathrm{H}^*(Z, K)$ is finite over the subalgebra of $\mathrm{H}^*(Z, \mathbf{Q}_\ell)$ generated by*

$$c_1(E), \dots, c_N(E),$$

where $c_i(E)$ is the ith Chern class of E, of cohomological degree $2i$. In particular, $\mathrm{H}^*(Z, \mathbf{Q}_\ell)$ is a finitely generated $\mathbf{Q}_\ell$-algebra.

Proof. Let $g : Z \to B\mathrm{GL}_N$ classify E, and put

$$Z_0 = Z \times_{B\mathrm{GL}_N} \operatorname{Spec} k.$$

The geometric stabilizers of Z_0 are the kernels in Definition 2.9. Lemma 2.8 shows that $\mathrm{R\Gamma}(Z_0, K|_{Z_0})$ is bounded with finite-dimensional cohomology. By smooth base change along $\operatorname{Spec} k \to B\mathrm{GL}_N$, this is the pullback of $\mathrm{R}g_* K$. Bounded constructibility can be checked on a smooth presentation, so $\mathrm{R}g_* K \in D^b_c(B\mathrm{GL}_N, \mathbf{Q}_\ell)$. This is a boundedness assertion about the sheaf complex; its global sections on $B\mathrm{GL}_N$ may be unbounded.

Every constructible sheaf on $B\mathrm{GL}_N$ is constant of finite rank. Indeed, its pullback to the smooth presentation $\operatorname{Spec} k \to B\mathrm{GL}_N$ is a finite-dimensional vector space; the descent isomorphism is constant on the connected scheme GL_N, and its restriction to the identity is the identity. The cohomology of $B\mathrm{GL}_N$ is

$$R := \mathrm{H}^*(B\mathrm{GL}_N, \mathbf{Q}_\ell) = \mathbf{Q}_\ell[c_1, \dots, c_N], \qquad |c_i| = 2i.$$

For finite coefficients this is [IZ16, Theorem 4.4]; the rational statement follows degree by degree by inverse limit and inversion of ℓ. In each fixed degree the groups with $\mathbf{Z}/\ell^a$-coefficients are free of fixed finite rank, with surjective transition maps.

The cohomology sheaves of $\mathrm{R}g_*K$ are finite direct sums of the constant sheaf and vanish outside a finite range. Apply $\mathrm{R}\Gamma(B\mathrm{GL}_N, -)$ to the canonical truncation triangles. Since R is noetherian, induction shows that $\mathrm{H}^*(B\mathrm{GL}_N, \mathrm{R}g_*K)$ is finite over R: at each step it is an extension of subquotients of two finite graded R-modules. The action factors through g^*, proving the assertion. □

Lemma 2.11 (Finite generation descends along module retractions)**.** *Let $A \hookrightarrow B$ be an inclusion of nonnegatively graded, graded-commutative $\mathbf{Q}_\ell$-algebras, with $A^0 = B^0 = \mathbf{Q}_\ell$. Suppose that B is finitely generated and that there is a degree-zero A-linear map*

$$r : B \longrightarrow A, \qquad r(1) = 1.$$

Then A is finitely generated.

Proof. The algebra B is finite over the subalgebra generated by finitely many homogeneous even generators: its odd generators square to zero. It is therefore noetherian for homogeneous ideals. Write $A_+ = \bigoplus_{n>0} A^n$. The ideal A_+B is generated by finitely many homogeneous elements $a_1, \dots, a_s$ of A_+.

For homogeneous $a \in A_+$, write $a = \sum_i a_i b_i$ with homogeneous coefficients of the required degrees. Applying r gives

$$a = \sum_i a_i r(b_i).$$

Every nonzero $r(b_i)$ has degree strictly smaller than a. Induction on degree proves that the a_i generate A over $\mathbf{Q}_\ell$. □

Construction 2.12 (Fundamental class)**.** Let Y be a reduced irreducible stack of dimension d. The fundamental class of Y gives a morphism

$$\mathrm{cl}_Y : \mathbf{Q}_{\ell,Y}(d)[2d] \longrightarrow \omega_Y.$$

Let $j : V \hookrightarrow Y$ be a dense smooth open with reduced complement Z. The usual dimension bound for the dualizing complex, checked on a smooth presentation, gives $\omega_Z \in D^{\geq -2\dim Z}$. Since $\dim Z < d$, localization therefore gives

$$\mathrm{H}^{-2d}(Y, \omega_Y(-d)) \xrightarrow{\sim} \mathrm{H}^{-2d}(V, \omega_V(-d)).$$

The orientation class of V extends uniquely across this isomorphism and defines cl_Y. The construction is compatible with smooth pullback, with the relative shift and Tate twist.

Proposition 2.13 (Trace retraction)**.** *Let k be an algebraically closed field and let ℓ be invertible in k. Let X and Y be quasi-separated reduced irreducible Artin stacks of finite type over k, and let $f : Y \to X$ be representable, proper and birational. Suppose that X is $\mathbf{Q}_\ell$-rationally smooth. The unit*

$$\mathbf{Q}_{\ell,X} \longrightarrow \mathrm{R}f_*\mathbf{Q}_{\ell,Y}$$

admits a retraction. On cohomology, it induces a degree-zero $\mathrm{H}^*(X, \mathbf{Q}_\ell)$-linear map

$$f_* : \mathrm{H}^*(Y, \mathbf{Q}_\ell) \longrightarrow \mathrm{H}^*(X, \mathbf{Q}_\ell), \qquad f_*f^* = \mathrm{id}.$$

If $\mathrm{H}^*(Y, \mathbf{Q}_\ell)$ is finitely generated, then so is $\mathrm{H}^*(X, \mathbf{Q}_\ell)$. Under this finite-generation hypothesis, if $K \in D^b_c(X, \mathbf{Q}_\ell)$ and $\mathrm{H}^*(Y, f^*K)$ is finite over $\mathrm{H}^*(Y, \mathbf{Q}_\ell)$, then $\mathrm{H}^*(X, K)$ is finite over $\mathrm{H}^*(X, \mathbf{Q}_\ell)$.

Proof. Let $d = \dim X = \dim Y$, and let cl_Y be the fundamental class from Construction 2.12.

Duality and rational smoothness give

$$f^!\omega_X = \omega_Y, \qquad \omega_X \simeq \mathbf{Q}_{\ell,X}(d)[2d].$$

Thus cl_Y induces a map $\mathbf{Q}_{\ell,Y} \to f^!\mathbf{Q}_{\ell,X}$. Since f is representable and proper, the counit defines

$$\mathrm{R}f_*\mathbf{Q}_{\ell,Y} \longrightarrow \mathrm{R}f_!f^!\mathbf{Q}_{\ell,X} \longrightarrow \mathbf{Q}_{\ell,X}. \tag{2.2}$$

The composite of the unit with (2.2) is the identity on a dense open over which f is an isomorphism and X is smooth. Since

$$\mathrm{Hom}(\mathbf{Q}_{\ell,X}, \mathbf{Q}_{\ell,X}) = \mathrm{H}^0(X, \mathbf{Q}_\ell) = \mathbf{Q}_\ell,$$

it is the identity on X. The projection formula yields

$$f_*(f^*a \smile b) = a \smile f_*b.$$

Thus f^* is injective and f_* is a graded module retraction. Both cohomology algebras are connected, so Lemma 2.11 proves the ring assertion.

For general K, put $A = \mathrm{H}^*(X, \mathbf{Q}_\ell)$, $B = \mathrm{H}^*(Y, \mathbf{Q}_\ell)$, $M = \mathrm{H}^*(X, K)$ and $N = \mathrm{H}^*(Y, f^*K)$. Tensoring (2.2) with K and applying the proper projection formula gives a retraction $r_K : N \to M$ satisfying

$$r_K(b \smile f^*m) = f_*(b) \smile m \qquad (b \in B,\ m \in M).$$

As in Lemma 2.11, B is finite over a noetherian even subalgebra. Hence the finite B-module N is noetherian, and its submodule $B \cdot f^*M$ is generated by finitely many homogeneous elements $f^*m_1, \dots, f^*m_s$. For $m \in M$, write $f^*m = \sum_j b_j \smile f^*m_j$. Applying r_K gives

$$m = \sum_j f_*(b_j) \smile m_j.$$

Thus $m_1, \dots, m_s$ generate M over A. □

Remark 2.14. The trace in Proposition 2.13 need not be multiplicative. A subalgebra of a finitely generated algebra need not be finitely generated; the module retraction, not the inclusion alone, is essential. Rational smoothness is used precisely in the identification of ω_X in that proof.

3. Vector bundles on proper modifications

Convention 3.1 (Projectivizations and projective morphisms)**.** For a coherent sheaf $\mathcal{F}$ on X, the projectivization $\mathbf{P}_X(\mathcal{F})$ parametrizes invertible quotients. We use projective morphisms in the sense of [Ryd16, Definition 8.5]: the ambient projectivization may be that of a coherent sheaf.

Definition 3.2 (Faithful vector bundles)**.** A vector bundle E on an Artin stack Z is *faithful* if, at every geometric point $z \to Z$, the homomorphism $G_z \to \mathrm{GL}(E_z)$ has trivial group-scheme kernel. For E of rank n, this is equivalent to representability of its classifying morphism $Z \to B\mathrm{GL}_n$.

Lemma 3.3 (Extending bundles by modification)**.** *Let X be reduced and irreducible, and let $\mathcal{F}$ be a coherent sheaf whose restriction to a dense open $U \subset X$ is locally free of rank $n \geq 1$. There are a representable projective birational morphism $f : Y \to X$, with Y reduced and irreducible and f an isomorphism over U, and a locally free quotient $f^*\mathcal{F} \twoheadrightarrow E$ of rank n extending $\mathcal{F}|_U$.*

Proof. Take the reduced closure of $U = \mathrm{Gr}_n(\mathcal{F})|_U$ in the Grassmannian of rank-n locally free quotients, with its universal quotient E. The closed Plücker embedding into $\mathbf{P}_X(\bigwedge^n \mathcal{F})$ makes f representable and projective. Over U a rank-n quotient of $\mathcal{F}$ is an isomorphism. □

Proof of Theorem C for quasi-affine diagonal. Choose a smooth surjection $p : U \to X$ with U an affine finite-type scheme. The morphism p is quasi-affine. Indeed, for an affine scheme $S \to X$, the morphism

$$U \times_X S \longrightarrow U \times_k S$$

is a base change of the quasi-affine diagonal of X, and $U \times_k S \to S$ is affine. In particular, p is quasi-projective. By [Ryd16, Theorem 8.6], there are a coherent sheaf $\mathcal{F}$ on X and a quasi-compact locally closed embedding

$$U \hookrightarrow \mathbf{P}_X(\mathcal{F}).$$

This gives an invertible sheaf $\mathcal{L}$ on U and a surjection $p^*\mathcal{F} \twoheadrightarrow \mathcal{L}$.

Generic freeness on a smooth presentation gives a dense open $X^\circ \subset X$ on which $\mathcal{F}$ is locally free, of rank $n \geq 1$. Apply Lemma 3.3 to obtain a projective birational map $f\colon Y \to X$ and write

$$0 \longrightarrow \mathcal{N} \longrightarrow f^*\mathcal{F} \longrightarrow E \longrightarrow 0$$

for the tautological quotient. The vector bundle E has rank n, and $\mathcal{N}$ vanishes over $Y^\circ = f^{-1}(X^\circ)$.

Set $U_Y = U \times_X Y$ and let $p_Y : U_Y \to Y$ be the projection. This is a smooth surjection from an algebraic space. Hence U_Y is reduced, and $U_Y^\circ = p_Y^{-1}(Y^\circ)$ is schematically dense: it is topologically dense because smooth morphisms are open. Let $\mathcal{L}_Y$ be the pullback of $\mathcal{L}$. The composite

$$p_Y^*\mathcal{N} \longrightarrow p_Y^* f^*\mathcal{F} \longrightarrow \mathcal{L}_Y$$

vanishes on U_Y°, and therefore vanishes everywhere. Indeed, an invertible sheaf on a reduced noetherian algebraic space has no nonzero section supported outside a dense open. We obtain a surjection $p_Y^*E \twoheadrightarrow \mathcal{L}_Y$, so the locally closed embedding

$$U_Y \hookrightarrow \mathbf{P}_Y(f^*\mathcal{F})$$

factors through the closed subspace $\mathbf{P}_Y(E)$. The induced morphism

$$U_Y \hookrightarrow \mathbf{P}_Y(E) \tag{3.1}$$

is again a locally closed embedding.

At a geometric point $y \to Y$, the group scheme G_y acts freely on the fiber U_y. This follows directly from the description $U_y = U \times_X y$ and the fact that U is an algebraic space. The locally closed embedding (3.1) is equivariant on fibers. The kernel of $G_y \to \mathrm{GL}(E_y)$ acts trivially on $\mathbf{P}(E_y)$, hence trivially on U_y, so it is trivial. After the faithfully flat base change $U_y \to y$, freeness identifies the stabilizer with the trivial group scheme.

Thus E is faithful. Its frame bundle $P = \mathrm{Fr}(E)$ is an algebraic space and $Y \simeq [P/\mathrm{GL}_n]$. Take $V = E$. If $\operatorname{char} k = 0$, a further projective resolution, functorial for smooth morphisms, makes Y smooth; the pullback of E remains faithful because the modification is representable. The smooth descent of this resolution is described in Construction 3.9 below. □

For the remaining case of Theorem C, assume that k has characteristic zero.

Lemma 3.4. *Let $h : \mathcal{W} \to P$ be a morphism of locally noetherian Artin stacks with relative inertia of finite presentation. Suppose that h induces an injection on stabilizer group schemes at a geometric point w. There is an open neighborhood $\mathcal{W}^\circ$ of w on which every geometric relative stabilizer of h is finite.*

Proof. Write $K = I_{\mathcal{W}/P}$ and let $e : \mathcal{W} \to K$ be the identity section. The coherent sheaf $e^*\Omega^1_{K/\mathcal{W}}$ has zero fiber at w, so it vanishes on a neighborhood by Nakayama's lemma. There every geometric relative stabilizer has zero cotangent space at its identity. Translation makes it a zero-dimensional reduced group algebraic space of finite type, hence a finite group scheme. □

Remark 3.5 (Finite relative stabilizers)**.** Lemma 3.4 asserts finiteness of the geometric fibers of the relative inertia, not that the relative inertia morphism is finite. It does not assert representability of h.

Lemma 3.6 (Boundary divisors detect stabilizer tori)**.** *Suppose that* $\operatorname{char} k = 0$. *Let P be a smooth Artin stack over k with affine stabilizers. Let $D_1, \ldots, D_m$ be smooth effective Cartier divisors on P, and suppose that*

$$P^\circ = P \setminus \bigcup_{i=1}^m D_i$$

is an algebraic space. Then $\bigoplus_i \mathcal{O}_P(D_i)$ detects stabilizer tori.

Proof. Suppose that a positive-dimensional torus $T \subset G_p$ at a geometric point p acts trivially on each fiber $\mathcal{O}_P(D_i)|_p$. We may extend the algebraically closed ground field so that p is rational. In characteristic zero G_p is smooth, and G_p/T is smooth. Since T is linearly reductive, [AHR20, Theorem 1.1] gives an affine scheme W, a T-action with fixed point w, and a smooth morphism

$$h : ([W/T], w) \longrightarrow (P, p)$$

inducing the prescribed inclusion $T \hookrightarrow G_p$. Apply Lemma 3.4 to h. After replacing W by the corresponding T-invariant open neighborhood of w, every geometric relative stabilizer of h is finite. The scheme W is smooth over k, since $W \to [W/T] \to P$ is smooth.

Let $\Delta_i \subset W$ be the pullback of D_i. It is a smooth T-invariant Cartier divisor, possibly empty. For $w \in \Delta_i$, the normal line $N_{\Delta_i/W,w}$ is the fiber of $\mathcal{O}_W(\Delta_i)$, and T acts trivially on it. Exactness of invariants for T gives a surjection

$$(T_w W)^T \longrightarrow N_{\Delta_i/W,w} \tag{3.2}$$

from the normal tangent sequence.

The fixed locus W^T is smooth and satisfies $T_w(W^T) = (T_w W)^T$; see [RY00, Lemma 4.2]. Let C be its irreducible component through w. If $w \in \Delta_i$, the surjection (3.2) shows that C is not contained in Δ_i. The same conclusion is immediate when $w \notin \Delta_i$. Thus there is a geometric point

$$w' \in C \setminus \bigcup_i \Delta_i.$$

Its stabilizer in $[W/T]$ is T. Its image lies in P°, whose stabilizers are trivial. Hence the relative stabilizer of h at w' is T, contradicting its finiteness. □

Remark 3.7 (Smoothness of boundary divisors)**.** A reduced divisor has *normal crossings* if its pullback to a smooth presentation is étale locally defined by a product of distinct local coordinates. It has *simple normal crossings* if, in addition, each irreducible component is a smooth effective Cartier divisor.

The individual divisors in Lemma 3.6 must be smooth. A divisor that has normal crossings on a presentation need not satisfy this requirement: descent can permute branches that meet. Lemma 3.8 separates these branches.

Lemma 3.8 (Separating boundary branches)**.** *Let Y be a smooth finite-type Artin stack and let $D \subset Y$ be a reduced normal-crossings divisor in the above sense. There is a finite sequence of blowups in smooth centers supported over D such that the resulting stack Y' is smooth and the reduced inverse image of D is a union of finitely many smooth effective Cartier divisors. The sequence is compatible with smooth base change, after omission of empty centers, and is an isomorphism over $Y \setminus D$.*

Proof. Follow [AT19, Remark 2.2.8(ii)], counting only branches of the strict transform of the original divisor D. If their maximum multiplicity is $m \geq 2$, blow up the reduced locus where m branches meet. In normal-crossings coordinates this center is $t_1 = \cdots = t_m = 0$, hence is smooth. The charts $t_i = t_j u_i$ for $i \neq j$ show that the maximum multiplicity strictly decreases and that normal crossings with the earlier exceptional divisors are preserved. The center meets those divisors transversely, so their strict transforms remain smooth; the new exceptional divisor is the projectivization of the normal bundle and is smooth.

Iteration makes the strict transform of D smooth. Together with the exceptional divisors labelled by their stages of creation, it gives the required smooth divisors, possibly disconnected. The centers are intrinsic and commute with smooth pullback, after omission of empty centers. Thus the construction descends to stacks and is unchanged off D. □

Construction 3.9 (Smooth descent of resolution of pairs)**.** Suppose that k has characteristic zero. Let Z be a quasi-separated reduced irreducible Artin stack of finite type over k with affine stabilizers, and let $U \subset Z$ be a smooth dense open. Functorial resolution of the pair $(Z, Z \setminus U)$ is supplied by [AT19, Theorem 2.2.10]. Its functoriality for smooth morphisms permits application to stacks: choose a smooth presentation $S \to Z$, apply the resolution to S and its pulled-back boundary, and use the canonical compatibility with the two projections $S \times_Z S \rightrightarrows S$. The resulting identifications satisfy the cocycle condition and descend the modification. If the groupoid is an algebraic space, first extend the resolution functor from schemes to algebraic spaces by étale descent. Smooth functoriality then gives the same descent along the presenting groupoid. No separatedness of the diagonal is used. See [AT19, §6.1] for this descent.

More precisely, use the finite blowup sequence in the proof of [AT19, Theorem 2.2.10]. At each stage, smooth functoriality identifies the two pullbacks of the coherent ideal defining the center on the presenting groupoid, compatibly with the cocycle, after omission of empty centers. Faithfully flat descent gives a coherent ideal $\mathcal{I}_i$ on the corresponding stack Z_i. Blowup commutes with flat base change, so the next stage is

$$Z_{i+1} = \operatorname{Proj}_{Z_i}\Big(\bigoplus_{n\geq 0} \mathcal{I}_i^n\Big) \hookrightarrow \mathbf{P}_{Z_i}(\mathcal{I}_i).$$

Each stage is therefore representable and projective, with its relatively ample tautological line bundle. Suitable tensor products of powers of their pullbacks give a relatively ample line bundle for the composite; see [AT19, §§2.1.8 and 2.1.10].

The result is a representable projective birational morphism $\widetilde{Z} \to Z$, unchanged over U, with $\widetilde{Z}$ smooth and reduced boundary a normal-crossings divisor.

Proof of Theorem C in characteristic zero. By [HR15, Proposition 2.6(1)], there is a dense open $U \subset X$ with a representable quasi-affine morphism to $B\mathrm{GL}_n$ for some $n \geq 1$. Pulling back the standard bundle gives a faithful vector bundle E_U. Shrinking U, we may assume that U is smooth over k.

Let $j : U \hookrightarrow X$. The quasi-coherent sheaf j_*E_U is the union of its coherent subsheaves, and quasi-compactness of U gives a coherent subsheaf $\mathcal{F}$ with $\mathcal{F}|_U = E_U$; see the coherent approximation statement before [HR15, Proposition 2.6]. By Lemma 3.3, there is a representable projective birational morphism $X_1 \to X$, unchanged over U, carrying a vector bundle E_1 extending E_U.

Apply Construction 3.9 to

$$(X_1, X_1 \setminus U).$$

We obtain a representable projective birational morphism $Y_0 \to X_1$, unchanged over U, with Y_0 smooth and reduced boundary a normal-crossings divisor. Apply Lemma 3.8. We may therefore write the resulting smooth modification as $Y \to X$ with

$$Y \setminus U = \bigcup_{i=1}^{m} D_i, \tag{3.3}$$

where every D_i is a smooth effective Cartier divisor. The stack Y remains quasi-separated with affine stabilizers: the modification is representable and proper, and the induced maps on stabilizers are injective.

Let E be the pullback of E_1 to Y. Consider its frame stack

$$P = \mathrm{Fr}(E) \longrightarrow Y.$$

It is a smooth affine GL_n-torsor, so P is smooth and quasi-separated with affine stabilizers. Over U it is an algebraic space, since E_U is faithful. The inverse images of the divisors in (3.3) are smooth Cartier divisors on P, with complement P_U.

Set

$$V = E \oplus \bigoplus_{i=1}^{m} \mathcal{O}_Y(D_i). \tag{3.4}$$

Suppose a positive-dimensional torus $T \subset G_y$ acts trivially on V_y. It acts trivially on E_y, hence fixes a frame. It is therefore a subgroup of the stabilizer of a point $p \in P$ over y, and acts trivially on all the pulled-back boundary line bundles. This contradicts Lemma 3.6. Thus V detects stabilizer tori. The Grassmannian modification is projective, and the subsequent modifications are sequences of blowups. Their composite $Y \to X$ is therefore projective. □

4. Extension algebras of intersection complexes

Convention 4.1 (Noetherianity)**.** Noetherianity refers to the underlying rings and modules, without commutativity or grading restrictions.

Lemma 4.2 (Noetherianity passes to idempotent corners)**.** *Let S be a ring and $e \in S$ an idempotent. If M is a noetherian right S-module, then Me is a noetherian right eSe-module. The analogous statement holds for left modules. In particular, a corner of a left and right noetherian ring is left and right noetherian.*

Proof. For an eSe-submodule $N \subset Me$, one has $(NS)e = N$: for $n = ne$, $nse = n(ese) \in N$, and the reverse inclusion follows from $ne = n$. An ascending chain of such submodules gives an ascending chain of S-submodules of M, so it stabilizes. For the ring assertion, apply this to the right S-module eS and the left S-module Se. □

Proposition 4.3 (Extension algebras of proper pushforwards)**.** *Let $f : Y \to X$ be representable and proper. Suppose that Y has affine stabilizers and a rank-N vector bundle V detecting stabilizer tori. For every $J \in D^b_c(Y, \mathbf{Q}_\ell)$, the algebra*

$$\mathrm{Ext}^*_X(\mathrm{R}f_*J, \mathrm{R}f_*J)$$

is left and right noetherian. For every $K \in D^b_c(X, \mathbf{Q}_\ell)$, the module $\mathrm{Ext}^*_X(\mathrm{R}f_*J, K)$ is noetherian on the right over this algebra, and $\mathrm{Ext}^*_X(K, \mathrm{R}f_*J)$ and $\mathrm{H}^*(X, \mathrm{R}f_*J \otimes^{\mathrm{L}} K)$ are noetherian on the left.

Proof. The Chern classes $c_i(V)$ act by graded endomorphisms of J and induce, after proper pushforward, a graded algebra homomorphism

$$\mathbf{Q}_\ell[t_1, \dots, t_N] \longrightarrow \mathrm{Ext}^*_X(\mathrm{R}f_*J, \mathrm{R}f_*J), \qquad |t_i| = 2i.$$

Its image need not be central. Adjunction and the projection formula give

$$\mathrm{RHom}_X(\mathrm{R}f_*J, K) \simeq \mathrm{R}\Gamma\big(Y, \mathrm{R}\mathcal{H}om(J, f^!K)\big), \tag{4.1}$$

$$\mathrm{RHom}_X(K, \mathrm{R}f_*J) \simeq \mathrm{R}\Gamma\big(Y, \mathrm{R}\mathcal{H}om(f^*K, J)\big), \tag{4.2}$$

$$\mathrm{R}\Gamma(X, \mathrm{R}f_*J \otimes^{\mathrm{L}} K) \simeq \mathrm{R}\Gamma(Y, J \otimes^{\mathrm{L}} f^*K). \tag{4.3}$$

All complexes inside $\mathrm{R}\Gamma(Y, -)$ are bounded constructible. Indeed, f is representable and proper, and internal Hom preserves bounded constructibility by the identity $\mathrm{R}\mathcal{H}om(A, B) \simeq D_Y(A \otimes^{\mathrm{L}} D_Y B)$, where D_Y denotes Verdier duality. Here duality preserves bounded constructibility by [LZ24, Propositions 7.4.2, 7.5.2 and 7.5.4] and the corresponding boundedness on a finite smooth presentation.

The Chern-class actions in (4.1) to (4.3) correspond respectively to precomposition, postcomposition, and action on the first tensor factor. By Proposition 2.10, the cohomology of each of these three complexes is finite over $\mathbf{Q}_\ell[t_1, \dots, t_N]$. Taking $K = \mathrm{R}f_*J$ in the first two identities makes $\mathrm{Ext}^*_X(\mathrm{R}f_*J, \mathrm{R}f_*J)$ finite on both sides over this polynomial ring. Every one-sided ideal of the extension algebra, and every submodule of one of these three modules over that algebra, is also a submodule over the polynomial ring on the same side. Noetherianity of the polynomial ring proves the assertions. □

Remark 4.4 (Work of Sun)**.** Sun proves decomposition for proper morphisms with finite relative diagonal between stacks with affine stabilizers over finite fields [Sun12, Theorem 1.2] and over $\mathbf{C}$ [Sun17, Theorem 4.2.4]. Below we use the results of Sun–Zheng over arbitrary fields [SZ16, §3.3] and smooth descent.

Lemma 4.5 (Splitting off the intersection complex)**.** *Let k be an algebraically closed field, with ℓ invertible in k. Let $f : Y \to X$ be a representable projective birational morphism of reduced irreducible finite-type Artin stacks with affine stabilizers. Then IC_X is a direct summand of $\mathrm{R}f_* \mathrm{IC}_Y$ in $D_c^b(X, \mathbf{Q}_\ell)$.*

Proof. Choose an f-ample invertible sheaf with first Chern class η. On smooth scheme presentations, intersection complexes are admissible semisimple, their proper direct images are semisimple, and relative hard Lefschetz holds by [SZ16, Remarks 3.3.3–3.3.4 and Proposition 3.3.10]. These assertions descend from $\overline{\mathbf{Q}}_\ell$ to $\mathbf{Q}_\ell$ by faithful extension of scalars. Smooth base change and compatibility of intersection complexes with smooth pullback therefore give

$$\eta^a : {}^p\mathcal{H}^{-a}(\mathrm{R}f_* \mathrm{IC}_Y) \xrightarrow{\sim} {}^p\mathcal{H}^a(\mathrm{R}f_* \mathrm{IC}_Y)(a) \qquad (a \geq 0). \tag{4.4}$$

After trivializing the Tate twist, Deligne's splitting criterion for the bounded perverse t-structure [Del68, Théorème 1.5] gives

$$\mathrm{R}f_* \mathrm{IC}_Y \simeq \bigoplus_a {}^p\mathcal{H}^a(\mathrm{R}f_* \mathrm{IC}_Y)[-a].$$

Choose a dense smooth open $U \subset X$ over which f is an isomorphism. On a smooth presentation $t : T \to X$ of pure relative dimension s, semisimplicity gives a summand IC_T of $t^{*p}\mathcal{H}^0(\mathrm{R}f_* \mathrm{IC}_Y)[s]$, with complement supported on $T \setminus U_T$. Its projector is canonical, since there are no morphisms in either direction between an intermediate extension from U_T and a perverse sheaf supported on its complement. The projectors therefore descend in the perverse heart [LO09, Proposition 7.1 and §8]. Their image is IC_X, which is thus a summand of ${}^p\mathcal{H}^0(\mathrm{R}f_* \mathrm{IC}_Y)$ and hence of $\mathrm{R}f_* \mathrm{IC}_Y$. ☐

Construction 4.6 (The summand corresponding to the intersection complex)**.** Let X satisfy the hypotheses of Theorem A, and let $i_a : X_a \hookrightarrow X$, for $1 \leq a \leq r$, be its reduced irreducible components. Each X_a inherits the stabilizer and diagonal hypotheses of X. Apply Theorem C to choose $f_a : Y_a \to X_a$ and a bundle V_a detecting stabilizer tori. The composite maps $i_a f_a$ assemble to a representable projective morphism

$$f : Y := \coprod_{a=1}^{r} Y_a \longrightarrow X.$$

Adding trivial summands to the V_a makes their ranks equal, so they define a bundle V on Y detecting stabilizer tori. By Lemma 4.5, each IC_{X_a} is a direct summand of $\mathrm{R}(f_a)_* \mathrm{IC}_{Y_a}$. Pushing forward by the i_a and taking their direct sum shows that IC_X is a direct summand of $\mathrm{R}f_* \mathrm{IC}_Y$. Let e be the corresponding idempotent endomorphism. Then

$$E_X \simeq e \operatorname{Ext}^*_X(\mathrm{R}f_* \mathrm{IC}_Y, \mathrm{R}f_* \mathrm{IC}_Y)e.$$

Proof of Theorem A. Choose the modification $f : Y \to X$ and the projector e onto IC_X as in Construction 4.6. Applying Proposition 4.3 with $J = \mathrm{IC}_Y$ and then Lemma 4.2 shows that E_X is left and right noetherian and that

$$\begin{aligned} \operatorname{Ext}^*_X(\mathrm{IC}_X, K) &\simeq \operatorname{Ext}^*_X(\mathrm{R}f_* \mathrm{IC}_Y, K)e, \\ \operatorname{Ext}^*_X(K, \mathrm{IC}_X) &\simeq e \operatorname{Ext}^*_X(K, \mathrm{R}f_* \mathrm{IC}_Y), \\ \mathrm{H}^*(X, \mathrm{IC}_X \otimes^{\mathrm{L}} K) &\simeq e\mathrm{H}^*(X, \mathrm{R}f_* \mathrm{IC}_Y \otimes^{\mathrm{L}} K) \end{aligned}$$

are noetherian, hence finite, as a right E_X-module in the first case and as left E_X-modules in the other two.

The summands $(i_a)_* \mathrm{IC}_{X_a}$ are pairwise nonisomorphic simple perverse sheaves, each with endomorphism ring $\mathbf{Q}_\ell$. Hence $E_X^0 = \mathbf{Q}_\ell^r$, and $E_X^n = 0$ for $n < 0$. Right noetherianity gives a finite generating set for the right ideal $E_X^{>0}$. Since this ideal is graded, replacing each generator by its finitely many homogeneous components gives a finite homogeneous generating set. Induction on degree shows that these, together with the component idempotents in E_X^0, generate E_X as a $\mathbf{Q}_\ell$-algebra. ☐

Corollary 4.7 (Finiteness under open restriction)**.** *Under the hypotheses of Theorem A, let $j : U \hookrightarrow X$ be an open embedding. Then restriction makes E_U finite as both a left and a right E_X-module.*

Proof. Since $j^* \mathrm{IC}_X = \mathrm{IC}_U$, adjunction identifies

$$E_U \simeq \mathrm{Ext}^*_X(\mathrm{IC}_X, \mathrm{R}j_* \mathrm{IC}_U) \simeq \mathrm{Ext}^*_X(j_! \mathrm{IC}_U, \mathrm{IC}_X).$$

The two complexes $\mathrm{R}j_* \mathrm{IC}_U$ and $j_! \mathrm{IC}_U$ are bounded constructible; for the former, this follows by smooth base change from the corresponding fact for finite-type schemes. Apply Theorem A. The resulting module structures are those induced by restriction $E_X \to E_U$. □

Remark 4.8 (Derived form)**.** With the data of Construction 4.6, let $g : Y \to B\mathrm{GL}_N$ classify V. The proof of Proposition 2.10 also shows that $\mathrm{R}\Gamma(Y, L)$ is a perfect dg module over $\mathrm{R}\Gamma(B\mathrm{GL}_N, \mathbf{Q}_\ell)$ for every $L \in D^b_c(Y, \mathbf{Q}_\ell)$. Indeed, the cohomology sheaves of $\mathrm{R}g_* L$ are finite constant sheaves and vanish outside a finite range. Its truncation triangles therefore express its global sections using finitely many shifts, sums and cones of this dg algebra, whose cohomology is $\mathbf{Q}_\ell[c_1, \ldots, c_N]$.

The identities (4.1) and (4.2) make $\mathrm{RHom}_X(\mathrm{R}f_* \mathrm{IC}_Y, \mathrm{R}f_* \mathrm{IC}_Y)$ perfect as both a left and a right dg module over $\mathrm{R}\Gamma(B\mathrm{GL}_N, \mathbf{Q}_\ell)$. Moreover, $\mathrm{RHom}_X(\mathrm{R}f_* \mathrm{IC}_Y, \mathrm{IC}_X)$ is a direct summand of the regular right module over this derived endomorphism algebra. The derived Yoneda functor $\mathrm{RHom}_X(\mathrm{R}f_* \mathrm{IC}_Y, -)$ is fully faithful on the thick subcategory generated by $\mathrm{R}f_* \mathrm{IC}_Y$, which contains IC_X as a retract; see [Kel06, Theorem 3.8(b)]. Thus $\mathrm{RHom}_X(\mathrm{IC}_X, \mathrm{IC}_X)$ is a derived idempotent corner of $\mathrm{RHom}_X(\mathrm{R}f_* \mathrm{IC}_Y, \mathrm{R}f_* \mathrm{IC}_Y)$. The characteristic-class action need not commute with the idempotent, so this does not assert that $\mathrm{RHom}_X(\mathrm{IC}_X, \mathrm{IC}_X)$ itself is perfect over $\mathrm{R}\Gamma(B\mathrm{GL}_N, \mathbf{Q}_\ell)$.

5. The rationally smooth case

Remark 5.1 (Connected rationally smooth components)**.** Passing to the reduction does not change étale cohomology or rational smoothness. A reduced connected rationally smooth stack X is irreducible. Indeed, put $d = \dim X$. Its intersection complex is the direct sum of the intersection complexes of its irreducible components, whereas $\mathbf{Q}_{\ell,X}[d]$ is indecomposable: its degree-zero endomorphism ring is $\mathrm{H}^0(X, \mathbf{Q}_\ell) = \mathbf{Q}_\ell$. Thus only one irreducible component can occur.

Proof of Corollary B. By Remark 5.1, we may pass to the reduction and work on one connected component, which is reduced and irreducible. Put $d = \dim X$. Rational smoothness gives $\mathrm{IC}_X \simeq \mathbf{Q}_{\ell,X}[d]$, hence

$$E_X = \mathrm{H}^*(X, \mathbf{Q}_\ell), \qquad \mathrm{H}^*(X, \mathrm{IC}_X \otimes^{\mathrm{L}} K[-d]) = \mathrm{H}^*(X, K).$$

The algebra and module actions are the usual cup-product actions under these identifications. Apply Theorem A with K replaced by $K[-d]$. There are only finitely many connected components, so the result follows for X. □

Remark 5.2 (A proof by trace)**.** The trace gives an alternative proof of Corollary B without decomposition. After the reduction in Remark 5.1, apply Theorem C to obtain a proper birational modification $f : Y \to X$ and a vector bundle V detecting stabilizer tori. By Proposition 2.10, the algebra $\mathrm{H}^*(Y, \mathbf{Q}_\ell)$ is finitely generated and $\mathrm{H}^*(Y, f^* K)$ is finite over it for every $K \in D^b_c(X, \mathbf{Q}_\ell)$. Both assertions then follow from Proposition 2.13.

Corollary 5.3 (Finiteness under proper pushforward)**.** *Let X satisfy the hypotheses of Corollary B, and let $h : Z \to X$ be representable and proper. For every $K \in D^b_c(Z, \mathbf{Q}_\ell)$, the graded module $\mathrm{H}^*(Z, K)$ is finite over $\mathrm{H}^*(X, \mathbf{Q}_\ell)$. In particular, $\mathrm{H}^*(Z, \mathbf{Q}_\ell)$ is finitely generated, without any rational-smoothness assumption on Z.*

Proof. Proper representable pushforward gives $\mathrm{R}h_* K \in D^b_c(X, \mathbf{Q}_\ell)$. Apply Corollary B to this complex. □

Remark 5.4 (Characteristic zero)**.** For quasi-affine diagonal, the geometric construction and the cohomological argument are independent of characteristic. With only affine stabilizers, the proof uses characteristic zero twice. First, it uses resolution and principalization of pairs functorial for smooth morphisms. This functoriality descends the construction from a presentation to the stack. Lemmas 3.4 and 3.8 do not require characteristic zero.

Second, in Lemma 3.6, characteristic zero ensures that G_p and G_p/T are smooth. The latter is the hypothesis needed for the smooth morphism supplied by [AHR20, Theorem 1.1]. Tori are linearly reductive in every characteristic, but smoothness of G_p/T requires a separate argument in positive characteristic.

Remark 5.5 (Generically finite traces)**.** A representable proper generically finite morphism $Y \to X$ of degree $e > 0$, with reduced irreducible source and rationally smooth target, gives a retraction after dividing its trace by e. Compatibility of fundamental classes with proper pushforward makes the composite with pullback multiplication by e.

6. Torus gerbes over nodal curves

Setup 6.1 (Monodromy data)**.** In this section k is algebraically closed of arbitrary characteristic, and ℓ is invertible in k. Fix a free abelian group L of rank $r \geq 1$ and automorphisms $\sigma_1, \dots, \sigma_m$, with $m \geq 1$. Put

$$\Gamma = \langle \sigma_1, \dots, \sigma_m \rangle, \qquad P = \mathrm{Sym}_{\mathbf{Q}_\ell}(L \otimes \mathbf{Q}_\ell).$$

Elements of L have cohomological degree two; P_n denotes polynomial degree n.

Construction 6.2 (A nodal curve and its character lattice)**.** Choose $2m$ distinct elements $a_i, b_i \in k$, $1 \leq i \leq m$, and set

$$R = \{ f \in k[t] : f(a_i) = f(b_i) \text{ for } 1 \leq i \leq m \}, \qquad C = \operatorname{Spec} R. \tag{6.1}$$

Let $\nu : \mathbf{A}^1_k \to C$ be induced by $R \subset k[t]$. Write $q_i \in C$ for the common image of a_i, b_i, and let $j_i : \operatorname{Spec} k \hookrightarrow C$ be its inclusion. On $C_{\text{ét}}$ define a sheaf of lattices

$$\mathcal{L} = \ker \left(\nu_* \underline{L}_{\mathbf{A}^1} \longrightarrow \bigoplus_{i=1}^{m} (j_i)_* \underline{L}_k \right), \tag{6.2}$$

where the ith component sends the two branch values to

$$(v_{a_i}, v_{b_i}) \longmapsto v_{a_i} - \sigma_i(v_{b_i}).$$

Lemma 6.3. *The ring R is a finitely generated integral k-algebra, and ν is the finite normalization of C. The points a_i, b_i map to a node q_i, and these m nodes are the only singularities of C. The sheaf $\mathcal{L}$ is étale locally constant and free of rank r. It is the character sheaf of a rank-r torus $T \to C$, and $\nu^* T$ is the split torus $T_0 = \mathbf{G}_{\mathrm{m}}^r$ over $\mathbf{A}^1$.*

Proof. Put $q(t) = \prod_{i=1}^{m} (t - a_i)(t - b_i)$. Then $k[q] \subset R \subset k[t]$. Since q is monic, $k[t]$ is finite over $k[q]$, and its $k[q]$-submodule R is finite. Thus R is of finite type over k, and $k[t]$ is finite over R. The elements q and tq belong to R, so $\operatorname{Frac} R = k(t)$. Since $k[t]$ is integrally closed, it is the normalization.

The ideal $qk[t]$ is contained in the conductor. Polynomial interpolation identifies

$$R/qk[t] \simeq k^m \subset k^{2m} \simeq k[t]/qk[t],$$

where each factor of k^m maps diagonally to the corresponding pair of factors. Thus ν is an isomorphism away from the $2m$ marked points and identifies precisely the stated pairs. Completion at q_i gives

$$\widehat{\mathcal{O}}_{C,q_i} \simeq \{ (f(u), g(v)) \in k[[u]] \oplus k[[v]] : f(0) = g(0) \} \simeq k[[u,v]]/(uv).$$

Away from the nodes the assertion about $\mathcal{L}$ is immediate. At a node, pass to an étale neighborhood on which its two branches are the two components of the normalization, and which contains no other node. Such a neighborhood exists from the étale local equation $uv = 0$ for an ordinary node. Changing basis on one component by σ_i identifies the relation in (6.2) with equality of branch values. The kernel is then the constant sheaf $\underline{L}$. Evaluation on either branch also gives $\nu^*\mathcal{L} \simeq \underline{L}_{\mathbf{A}^1}$.

Over an arbitrary base scheme, tori are anti-equivalent to étale locally constant free character lattices of finite rank. This applies to the nonnormal curve C and gives

$$T = \underline{\mathrm{Hom}}(\mathcal{L}, \mathbf{G}_{\mathrm{m}}).$$

Equivalently, trivialize $\mathcal{L}$ étale locally and descend $\mathbf{G}_{\mathrm{m}}^r$ using the resulting change-of-basis automorphisms. Effective descent for affine schemes makes T an affine group scheme, and smoothness and finite presentation descend as well. Its pullback to the normalization is split by the preceding identification of lattices. □

Notation 6.4 (Branch automorphisms)**.** Write $\alpha_i : T_0 \to T_0$ for the automorphism with $\alpha_i^* = \sigma_i$.

Construction 6.5 (The torus gerbe)**.** Set

$$X = B_C T, \qquad \widetilde{X} = X \times_C \mathbf{A}^1 \simeq \mathbf{A}^1 \times BT_0, \qquad Z_i = X \times_C q_i \simeq BT_0.$$

The stack X has affine diagonal: its relative diagonal over the affine curve C is a T-torsor. The morphism $C \to X$ is a smooth surjective presentation. Denote by

$$f : \widetilde{X} \to X, \qquad i_i : Z_i \hookrightarrow X$$

the natural morphisms.

Proof of Theorem D. Normalization gives the exact sequence

$$0 \longrightarrow \mathbf{Q}_{\ell,X} \longrightarrow f_*\mathbf{Q}_{\ell,\widetilde{X}} \longrightarrow \bigoplus_{i=1}^{m} (i_i)_*\mathbf{Q}_{\ell,Z_i} \longrightarrow 0. \tag{6.3}$$

This can be checked on the smooth presentation $C \to X$: away from the nodes f is an isomorphism, and at a node the stalk sequence is

$$0 \longrightarrow \mathbf{Q}_\ell \xrightarrow{a \mapsto (a,a)} \mathbf{Q}_\ell^2 \xrightarrow{(a,b) \mapsto b-a} \mathbf{Q}_\ell \longrightarrow 0.$$

Since f is finite and representable, proper base change gives no higher direct images. These assertions may first be checked with $\mathbf{Z}/\ell^a$-coefficients and then passed to adic coefficients.

The cohomology of BT_0 and homotopy invariance give

$$\mathrm{H}^*(\widetilde{X}, \mathbf{Q}_\ell) = \mathrm{H}^*(BT_0, \mathbf{Q}_\ell) = P, \qquad \mathrm{H}^*(Z_i, \mathbf{Q}_\ell) = P.$$

Identify Z_i using the first branch. The two restrictions to this fiber differ by $\alpha_i^* = \sigma_i$, so the map $(a, b) \mapsto b - a$ induces $\sigma_i - 1$. The long exact sequence of (6.3) is therefore

$$\begin{gathered} 0 \longrightarrow \mathrm{H}^{2n}(X, \mathbf{Q}_\ell) \longrightarrow P_n \xrightarrow{d_n} P_n^{\oplus m} \longrightarrow \mathrm{H}^{2n+1}(X, \mathbf{Q}_\ell) \longrightarrow 0, \\ n \geq 0, \end{gathered} \tag{6.4}$$

with $d(p) = ((\sigma_i - 1)p)_{i=1}^m$.

Restriction is multiplicative and identifies even cohomology with $A = P^\Gamma$. The connecting maps identify odd cohomology with $M = \operatorname{coker} d$ as an A-module: invariant polynomials have identical branch restrictions, so the action is induced by multiplication on $P^{\oplus m}$. Every odd class restricts to zero on $\widetilde{X}$. A product of two such classes has even degree and restricts to zero, hence vanishes by injectivity in even degree. Thus $\mathrm{H}^*(X, \mathbf{Q}_\ell) = A \oplus M$ with $M^2 = 0$. Since $A \subset P$ is reduced, the nilradical is exactly M, and

$$\mathrm{H}^*(X, \mathbf{Q}_\ell)/\sqrt{(0)} \simeq P^\Gamma. \tag{6.5}$$

Each cohomology group is finite-dimensional. More precisely, (6.4) gives

$$\dim_{\mathbf{Q}_\ell} M_n = (m-1)\binom{n+r-1}{r-1} + \dim_{\mathbf{Q}_\ell} A_n. \tag{6.6}$$

If Γ is finite, A is finitely generated and P is finite over A; see the proof of Proposition 6.7. Since d is A-linear, M is finite over A, and $A \oplus M$ is finitely generated.

Conversely, suppose $m \geq 2$ and $A \oplus M$ is finitely generated. As $M^2 = 0$, the odd members of a finite homogeneous generating set generate M over A. In polynomial degree, put

$$A_+ = \bigoplus_{n>0} A_n, \qquad Q = P/A_+P, \qquad W = M/A_+M.$$

Then W is finite-dimensional. Tensoring $P \xrightarrow{d} P^{\oplus m} \to M \to 0$ over A with $A/A_+ = \mathbf{Q}_\ell$ gives

$$Q \xrightarrow{\bar{d}} Q^{\oplus m} \longrightarrow W \longrightarrow 0, \qquad \dim W_n \geq (m-1)\dim Q_n.$$

Thus Q is finite-dimensional. Lifts of a homogeneous basis of Q generate P over A, by induction on degree: the remainder lies in A_+P, where the factors from P have smaller degree. Hence P is finite over A. The faithful action of Γ embeds it into the finite automorphism group of $\mathrm{Frac}(P)/\mathrm{Frac}(A)$, so Γ is finite. □

Example 6.6 (A single node with infinite monodromy)**.** The condition $m \geq 2$ in the finite-generation criterion is necessary. For $m = 1$, take

$$\sigma(x) = 2x + y, \qquad \sigma(y) = x + y.$$

This automorphism has distinct reciprocal eigenvalues, neither a root of unity, and therefore has infinite order. After a finite extension of $\mathbf{Q}_\ell$, invariant monomials are precisely the powers of the product of eigenvectors, so $P^\sigma = \mathbf{Q}_\ell[q]$ with $q = x^2 - xy - y^2$. Semisimplicity in each degree gives $P = \ker(\sigma - 1) \oplus (\sigma - 1)P$ as a P^σ-module. Thus $M \simeq P^\sigma$, and the cohomology algebra is $\mathbf{Q}_\ell[q] \otimes \bigwedge(\eta)$, with $|q| = 4$ and $|\eta| = 1$.

Proposition 6.7 (Finite generation with finite coefficients)**.** *For every stack X constructed in Theorem D and every integer $a \geq 1$, the algebra $\mathrm{H}^*(X, \mathbf{Z}/\ell^a)$ is finitely generated over $\mathbf{Z}/\ell^a$.*

Proof. Set $\Lambda = \mathbf{Z}/\ell^a$. The normalization argument applies directly with these coefficients and with $P_\Lambda = \mathrm{Sym}_\Lambda(L \otimes \Lambda)$. The automorphisms have finite image $\Gamma_a \subset \mathrm{GL}_r(\Lambda)$. Let $A_\Lambda = P_\Lambda^{\Gamma_a}$.

Finite generation here does not require division by $|\Gamma_a|$. Let $S \subset A_\Lambda$ be the Λ-subalgebra generated by the coefficients of the finitely many polynomials

$$\prod_{\gamma \in \Gamma_a} (T - \gamma z),$$

where z ranges over a basis of L. Each variable is integral over S, so P_Λ is finite over the noetherian algebra S. Its S-submodule A_Λ is finite as well. Thus A_Λ is a finitely generated noetherian Λ-algebra, and P_Λ is finite over it. The cokernel of $P_\Lambda \to P_\Lambda^{\oplus m}$ is consequently a finite A_Λ-module. The even and odd parts of cohomology are therefore finite over A_Λ, proving the assertion. □

Remark 6.8 (Dependence on a)**.** The proof of Proposition 6.7 supplies generators separately for each a; in general, it gives no uniform bound on their degrees as a varies. Consequently, finite generation for every $\mathbf{Z}/\ell^a$ does not supply a finite generating set after passing degreewise to $\mathbf{Z}_\ell$ and inverting ℓ.

We now specialize to Totaro's representation.

Construction 6.9 (Totaro's representation)**.** Put

$$B = (b_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{6.7}$$

Let L be the free abelian group with basis $x_1, \ldots, x_8, y_1, \ldots, y_8$. Define four commuting automorphisms by

$$\sigma_i(x_j) = x_j, \qquad \sigma_i(y_j) = y_j + b_{ij}x_j \quad (1 \leq i \leq 4,\ 1 \leq j \leq 8). \tag{6.8}$$

They are unipotent automorphisms. Write $\Gamma = \langle \sigma_1, \ldots, \sigma_4 \rangle$ and

$$P = \mathrm{Sym}_{\mathbf{Q}_\ell}(L \otimes \mathbf{Q}_\ell) = \mathbf{Q}_\ell[x_1, \ldots, x_8, y_1, \ldots, y_8].$$

Each variable has cohomological degree two; P_n denotes polynomial degree n.

Lemma 6.10. *The invariant ring P^Γ is not finitely generated over $\mathbf{Q}_\ell$.*

Proof. Let $U = \mathbf{G}^4_{\mathrm{a},\mathbf{Q}_\ell}$ act on P by

$$x_j \longmapsto x_j, \qquad y_j \longmapsto y_j + \left(\sum_{i=1}^{4} s_i b_{ij}\right) x_j,$$

where $(s_1, \ldots, s_4)$ are the coordinates on U. This is the representation of [Tot08, Corollary 7.1], which asserts that P^U is not finitely generated over any field of characteristic different from two. It applies over $\mathbf{Q}_\ell$, which has characteristic zero.

The four automorphisms (6.8) are the actions of the standard basis vectors of $\mathbf{Z}^4 \subset U(\mathbf{Q}_\ell)$. The set $\mathbf{Z}^4 \subset U(\mathbf{Q}_\ell)$ is Zariski dense: a polynomial vanishing on this set is zero by successive application of the fact that a nonzero one-variable polynomial has finitely many roots. For each $f \in P$, the condition that a point of U fix f is closed. It follows that $P^\Gamma = P^U$, proving the claim. ☐

Proof of Corollary E. Use the curve and torus of Theorem D for the four automorphisms (6.8). By (6.5), the reduced cohomology algebra is P^Γ, which is not finitely generated by Lemma 6.10. The assertions about the nilradical and individual cohomology groups follow from Theorem D. Finite generation with $\mathbf{Z}/\ell^a$-coefficients follows from Proposition 6.7. All geometric constructions are independent of ℓ. ☐

7. The affine-stabilizer hypothesis

The affine-stabilizer hypothesis cannot be omitted, even if the diagonal is proper.

Example 7.1 (A constant elliptic stabilizer)**.** Let E_0 be an elliptic curve over $\mathbf{C}$. For every prime ℓ,

$$\mathrm{H}^*(BE_0, \mathbf{Q}_\ell) = \mathbf{Q}_\ell[u, v], \qquad |u| = |v| = 2.$$

This is the calculation for classifying spaces used in the proof of Theorem 7.2 below, applied to a constant elliptic curve.

Theorem 7.2 (The Legendre elliptic gerbe)**.** *Let $S = \mathbf{P}^1_{\mathbf{C}} \setminus \{0, 1, \infty\}$, let $f : E \to S$ be the Legendre elliptic scheme*

$$y^2 z = x(x - z)(x - tz),$$

with its section at infinity, and put $\mathcal{X} = B_S E$. Then $\mathcal{X}$ is smooth and of finite type over $\mathbf{C}$, with proper diagonal. For every prime ℓ,

$$\mathrm{H}^*(\mathcal{X}, \mathbf{Q}_\ell) = \mathbf{Q}_\ell \oplus J, \qquad J^2 = 0,$$

where J is infinite-dimensional. In particular, this cohomology algebra is not finitely generated.

Proof. The presentation $S \to B_S E$ is smooth and surjective. The relative diagonal over S is an E-torsor, hence proper, and the diagonal of $S/\mathbf{C}$ is a closed embedding. Thus the absolute diagonal of $\mathcal{X}$ is proper.

Write $\pi : \mathcal{X} \to S$ for the projection and set $\mathcal{V} = \mathrm{R}^1 f_* \mathbf{Q}_\ell$. The calculation for classifying spaces gives

$$\mathrm{R}^{2n}\pi_*\mathbf{Q}_\ell \simeq \mathrm{Sym}^n \mathcal{V}, \qquad \mathrm{R}^{2n+1}\pi_*\mathbf{Q}_\ell = 0 \quad (n \geq 0). \tag{7.1}$$

Indeed, analytically the elliptic scheme is locally a bundle of real two-dimensional tori. Their classifying spaces are $K(\mathbf{Z}^2, 2)$, whose cohomology is the symmetric algebra on the first cohomology of the torus, placed in degree two. This identification is natural under monodromy. Apply étale–classical comparison [SGA4, Exposé XVI, Théorème 4.1] levelwise to the simplicial presentation $E^{\times_S \bullet}$ and use cohomological descent to obtain (7.1). The argument may first be made with $\mathbf{Z}/\ell^a$ and then passed degreewise to $\mathbf{Q}_\ell$-coefficients.

The space $S(\mathbf{C})$ is homotopy equivalent to a wedge of two circles. The Leray spectral sequence therefore has only columns zero and one and gives

$$\begin{aligned} \mathrm{H}^{2n}(\mathcal{X}, \mathbf{Q}_\ell) &\simeq \mathrm{H}^0(S, \operatorname{Sym}^n \mathcal{V}), \\ \mathrm{H}^{2n+1}(\mathcal{X}, \mathbf{Q}_\ell) &\simeq \mathrm{H}^1(S, \operatorname{Sym}^n \mathcal{V}). \end{aligned} \tag{7.2}$$

Up to a choice of basis and the isomorphism with the dual representation, the two monodromy generators on $\mathcal{V}$ are

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix};$$

see [Sti15, Lemma 16]. Their common invariants on $W_n = \operatorname{Sym}^n(\mathbf{Q}_\ell^2)$ vanish for $n > 0$. In fact, the fixed space of each nontrivial unipotent on W_n is the nth power of its fixed line, and the two lines here are distinct.

The cohomology of the corresponding local system is computed by

$$W_n \xrightarrow{v \mapsto ((A-1)v, (B-1)v)} W_n \oplus W_n,$$

where A and B denote their induced actions on W_n. For $n > 0$ this map is injective and has cokernel of dimension $n+1$. For $n = 0$ it is zero. Therefore

$$\begin{aligned} \mathrm{H}^0(\mathcal{X}, \mathbf{Q}_\ell) = \mathbf{Q}_\ell, \qquad & \mathrm{H}^1(\mathcal{X}, \mathbf{Q}_\ell) \simeq \mathbf{Q}_\ell^2, \\ \mathrm{H}^{2n}(\mathcal{X}, \mathbf{Q}_\ell) = 0, \qquad & \dim_{\mathbf{Q}_\ell} \mathrm{H}^{2n+1}(\mathcal{X}, \mathbf{Q}_\ell) = n+1 \quad (n \geq 1). \end{aligned} \tag{7.3}$$

Every product of positive-degree classes has positive even degree and hence vanishes. Taking $J = \mathrm{H}^{>0}(\mathcal{X}, \mathbf{Q}_\ell)$ proves the assertion. □

Remark 7.3 (Varying elliptic stabilizers)**.** Since $\mathcal{X}$ in Theorem 7.2 is smooth, $\mathrm{IC}_{\mathcal{X}} = \mathbf{Q}_{\ell,\mathcal{X}}[\dim \mathcal{X}]$ and

$$E_{\mathcal{X}} = \mathrm{H}^*(\mathcal{X}, \mathbf{Q}_\ell).$$

This algebra is not finitely generated; it is also nonnoetherian, since its infinite-dimensional square-zero ideal J is not finitely generated as a module. The affine-stabilizer hypothesis in Theorem A therefore cannot be omitted.

The reduced cohomology algebra in Theorem 7.2 is $\mathbf{Q}_\ell$. The example in Corollary E has affine diagonal and its reduced cohomology algebra is not finitely generated. The failure above comes from monodromy of the varying elliptic stabilizers; compare Example 7.1.

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