GALOIS REPRESENTATIONS ATTACHED TO CUSPIDAL NEWFORMS OF WEIGHT $k \geqslant 2$

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Abstract. We provide an account of Deligne's construction of a two-dimensional ℓ-adic Galois representation attached to a normalized cuspidal newform of arbitrary weight $k \geqslant 2$.

1. The theorem

Theorem 1.1 (Deligne [De71])**.** *Let ℓ be a prime and f be a normalized cuspidal newform of weight $k \geqslant 2$ and level N with Fourier coefficients a_n and Nebentypus χ. Then there exists a semisimple Galois representation $\rho_{f,\ell}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ which is unramified at all primes p not dividing ℓN and Frob_p, any arithmetic Frobenius over p, has characteristic polynomial $X^2 - a_p X + p^{k-1}\chi(p)$.*

Brauer-Nesbitt theorem from representation theory tells us that semisimple representations are uniquely determined upto isomorphism by their characteristic polynomials. Together with this and Chebotarev density theorem and some continuity arguments, it can be shown that the equality

$$\det(XI_2 - \mathrm{Frob}_p) = X^2 - a_p X + p^{k-1}\chi(p), \quad \text{for all } p \nmid \ell N,$$

determines $\rho_{f,\ell}$, if exists, upto isomorphism.

2. Hodge filtration and the Kodaira-Spencer map

Definition 2.1. Let S be a scheme. A smooth proper morphism $\mathscr{E} \to S$ with geometrically connected one-dimensional fibers of genus 1 and a specified section $e\colon S \to \mathscr{E}$ is called a **family of elliptic curves over** S**.**

It can be proven that $\mathscr{E}$ has the structure of a commutative group S-scheme which restricts to the usual group law of elliptic curves on geometric fibers. In the complex analytic setting, we replace the word "scheme" by "analytic space". Note that e is a closed embedding.

Let $f\colon \mathscr{E} \to S$ be a complex analytic family of elliptic curves. We have the de Rham exact sequence $0 \to f^{-1}\mathscr{O}_{\mathscr{E}} \to \mathscr{O}_{\mathscr{E}} \to \Omega^1_{\mathscr{E}/S} \to 0$. Taking pushforwards,

$$0 \to f_* f^{-1}\mathscr{O}_S \to f_*\mathscr{O}_{\mathscr{E}} \to f_*\Omega^1_{\mathscr{E}/S} \to \mathrm{R}^1 f_*(f^{-1}\mathscr{O}_S) \to \mathrm{R}^1 f_*\mathscr{O}_{\mathscr{E}} \to \mathrm{R}^1 f_*\Omega^1_{\mathscr{E}/S} \to \mathrm{R}^2 f_*(f^{-1}\mathscr{O}_S) \to \cdots$$

By Stein factorisation and Zariski's main theorem, $\mathscr{O}_S \to f_*\mathscr{O}_{\mathscr{E}}$ is an isomorphism. We have the natural isomorphisms

$$\mathrm{R}^i f_*\mathbb{Z} \otimes_{\mathbb{Z}} \mathscr{O}_S \cong \mathrm{R}^i f_*\mathbb{Z} \otimes_{\mathbb{Z}} f_* f^{-1}\mathscr{O}_S \cong \mathrm{R}^i f_*(f^{-1}\mathscr{O}_S).$$

So, the long exact sequence can be rewritten as

$$0 \to \mathrm{R}^1 f_*\mathscr{O}_{\mathscr{E}} \to \mathrm{R}^1 f_*\mathbb{Z} \otimes_{\mathbb{Z}} \mathscr{O}_S \to f_*\Omega^1_{\mathscr{E}/S} \to \mathrm{R}^1 f_*\Omega^1_{\mathscr{E}/S} \to \mathrm{R}^2 f_*\mathbb{Z} \otimes_{\mathbb{Z}} \mathscr{O}_S \to \cdots$$

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The map $\mathrm{R}^1 f_* \Omega^1_{\mathscr{E}/S} \to \mathrm{R}^2 f_* \mathbb{Z} \otimes_{\mathbb{Z}} \mathscr{O}_S$ is an isomorphism due to $\mathrm{R}^2 f_* \mathbb{Z} \cong \mathbb{Z}$ and Serre duality on fibers (proper base change). Writing $\omega = f_* \Omega^1_{\mathscr{E}/S}$, we have an exact sequence

$$0 \to \omega \to \mathrm{R}^1 f_* \mathbb{Z} \otimes_{\mathbb{Z}} \mathscr{O}_S \to \omega^\vee \to 0, \tag{1}$$

where $f_* \Omega^1_{\mathscr{E}/S} \cong \omega^\vee$ is due to Grothendieck duality. The above exact sequence is called the **Hodge filtration**. For any integer n, there is a cup product pairing of vector bundles

$$\mathrm{R}^1 f_* ((\Omega^1_{\mathscr{E}/S})^{\otimes n}) \otimes f_* ((\Omega^1_{\mathscr{E}/S})^{\otimes(1-n)}) \to \mathrm{R}^1 f_* \Omega^1_{\mathscr{E}/S}.$$

This pairing is perfect as seen be applying Serre duality on fibers. Putting $n = -1$, it follows that $\mathrm{R}^1 f_* (\Omega^\vee_{\mathscr{E}/S})$ is $\mathscr{O}_S$-dual to $f_* ((\Omega^1_{\mathscr{E}/S})^{\otimes 2})$. Now consider the dualized cotangent exact sequence

$$0 \to \Omega^\vee_{\mathscr{E}/S} \to \Omega^\vee_{\mathscr{E}} \to (f^* \Omega^1_S)^\vee \to 0.$$

Taking derived pushforwards gives a coboundary map $f_* ((f^* \Omega^1_S)^\vee) \to \mathrm{R}^1 f_* (\Omega^\vee_{\mathscr{E}/S})$. Dualizing this and using $f_* ((f^* \Omega^1_S)^\vee) \simeq \Omega^\vee_S$ (because $\mathscr{O}_S \simeq f_* \mathscr{O}_{\mathscr{E}}$), we get a map $f_* ((\Omega^1_{\mathscr{E}/S})^{\otimes 2}) \to \Omega^1_S$. By checking on fibers, we see that the natural map $\omega^{\otimes 2} \to f_* ((\Omega^1_{\mathscr{E}/S})^{\otimes 2})$ is an isomorphism. Therefore, we obtain a $\mathscr{O}_S$-linear map

$$\mathrm{KS}_{\mathscr{E}/S} \colon \omega^{\otimes 2} \to \Omega^1_X,$$

which we call the **Kodaira-Spencer map**. Since all of our steps behave well with base-change, we remark that $\mathrm{KS}_{\mathscr{E}/S}$ is compatible with base-change $S' \to S$ for S' smooth.

3. The Eichler-Shimura isomorphism

Let $\mathfrak{h}$ be the complex upper-half plane. Define the map of family of elliptic curves $f \colon \mathscr{E} \to \mathfrak{h}$ as the first projection of

$$\mathscr{E} = \{(z, [x:y:w]) \in \mathfrak{h} \times \mathbb{CP}^2 \colon y^2 w = 4x^3 - g_2(z) x w^2 - g_3(z) w^3\},$$

where the identity section is given by $e(z) = (z, [0:1:0])$. See [DS05, Section 1.4] for definitions of g_2 and g_3. Define $\Gamma = \Gamma(N) = \mathrm{Ker}(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$. Denote $Y_\Gamma = \mathfrak{h}/\Gamma$. Also, denote by X_Γ the compactification of Y_Γ. We state without proof the following fact–

Theorem 3.1. *Y_Γ represents the moduli functor* $\mathsf{AnaSp} \to \mathsf{Set}$ *which associates to an analytic space S the set of all isomorphism classes of complex analytic families of elliptic curves $\mathscr{E} \to S$ together with a pair of sections $P, Q \in \mathscr{E}(S)[N]$ such that $(P, Q) \colon (\mathbb{Z}/N\mathbb{Z})^{\oplus 2}_S \to \mathscr{E}[N]$ is an isomorphism of analytic group S-objects.*

Let $f_\Gamma \colon \mathscr{E}_\Gamma \to Y_\Gamma$ be the corresponding universal family of elliptic curves with "Γ-structure". This f_Γ is a "descent" of $f \colon \mathscr{E} \to \mathfrak{h}$. It is worth nothing that even though $\mathfrak{h}$ cannot be "algebraized", Y_Γ can be. It's clear that we have an exact sequence similar to (1)–

$$0 \to \omega_\Gamma \to \mathrm{R}^1 f_{\Gamma,*} \mathbb{Z} \otimes_{\mathbb{Z}} \mathscr{O}_{Y_\Gamma} \to \omega^\vee_\Gamma \to 0$$

This is a sequence of locally free sheaves as the following lemma and Grauert's theorem [MO173177] shows.

Lemma 3.2. $\mathrm{R}^1 f_* \mathbb{Z} \cong \mathbb{Z}^{\oplus 2}$.

Proof. We know that $\mathrm{R}^1 f_* \mathbb{Z}$ is the sheafification of $U \mapsto \mathrm{H}^1(f^{-1}(U), \mathbb{Z})$. Take some "small enough" simply-connected open subset $U \subset \mathfrak{h}$. By smoothness, $f^{-1}(U)$ is homeomorphic to $U \times \mathscr{E}_x$ where x is some point in U and $\mathscr{E}_x = f^{-1}(x)$. It is easily seen that $\mathrm{H}^1(U \times \mathscr{E}_x, \mathbb{Z}) \cong \mathrm{H}^1(\mathscr{E}_x, \mathbb{Z}) \cong \mathbb{Z}^{\oplus 2}$. Hence, the lemma follows. □

Let us write $\mathscr{U} = \mathrm{R}^1 f_{\Gamma,*}\mathbb{Z}$ and $\mathscr{U}^k = \mathrm{Sym}^k_{\mathbb{Z}}(\mathrm{R}^1 f_{\Gamma,*}\mathbb{Z})$. Since the above is an exact sequence of locally free sheaves, we have an injective map of $\mathscr{O}_{Y_\Gamma}$-modules $\omega_\Gamma^{\otimes k} \to \mathscr{U}^k \otimes_{\mathbb{Z}} \mathscr{O}_{Y_\Gamma}$. It is this map which will allow us to relate modular forms with étale cohomology. Tensoring by $\Omega^1_{Y_\Gamma}$, we get an injection

$$\omega_\Gamma^{\otimes k} \otimes \Omega^1_{Y_\Gamma} \to \mathscr{U}^k \otimes_{\mathbb{Z}} \Omega^1_{Y_\Gamma}.$$

Tensoring the de Rham exact sequence with $\mathscr{U}^k$ gives

$$0 \to \mathscr{U}^k \otimes_{\mathbb{Z}} \mathbb{C} \to \mathscr{U}^k \otimes_{\mathbb{Z}} \mathscr{O}_{Y_\Gamma} \to \mathscr{U}^k \otimes_{\mathbb{Z}} \Omega^1_{Y_\Gamma} \to 0.$$

We therefore have a $\mathbb{C}$-linear map

$$\delta\colon \mathrm{H}^0(Y_\Gamma, \Omega^1_{Y_\Gamma} \otimes_{\mathbb{Z}} \omega_\Gamma^k) \to \mathrm{H}^1(Y_\Gamma, \mathscr{U}_\Gamma^k \otimes_{\mathbb{Z}} \mathbb{C}).$$

We also have the "complex conjugate" map

$$\overline{\delta}\colon \overline{\mathrm{H}^0(Y_\Gamma, \Omega^1_{Y_\Gamma} \otimes_{\mathbb{Z}} \omega_\Gamma^k)} \to \mathrm{H}^1(Y_\Gamma, \mathscr{U}_\Gamma^k \otimes_{\mathbb{Z}} \mathbb{C}).$$

Define

$$\mathrm{sh}_\circ = \delta \oplus \overline{\delta}\colon \mathrm{H}^0(Y_\Gamma, \Omega^1_{Y_\Gamma} \otimes_{\mathbb{Z}} \omega_\Gamma^k) \oplus \overline{\mathrm{H}^0(Y_\Gamma, \Omega^1_{Y_\Gamma} \otimes_{\mathbb{Z}} \omega_\Gamma^k)} \to \mathrm{H}^1(Y_\Gamma, \mathscr{U}_\Gamma^k \otimes_{\mathbb{Z}} \mathbb{C}).$$

For any cohomology theory H whose variant with compact supports is denoted by H_c, denote $\widetilde{\mathrm{H}}^\bullet = \mathrm{Im}(\mathrm{H}_c^\bullet \to \mathrm{H}^\bullet)$ and $\widetilde{\mathrm{R}}^\bullet = \mathrm{Im}(\mathrm{R}_c^\bullet \to \mathrm{R}^\bullet)$. The invertible sheaf ω_Γ can be extended to X_Γ to a bigger invertible sheaf, which we denote by the same symbol ω_Γ.

Theorem 3.3 (Eichler-Shimura)**.** *There is an isomorphism* sh *such that the following diagram commutes.*

$$\begin{array}{ccc}
\mathrm{H}^0(X_\Gamma, \Omega^1_{X_\Gamma} \otimes_{\mathbb{Z}} \omega_\Gamma^k) \oplus \overline{\mathrm{H}^0(X_\Gamma, \Omega^1_{X_\Gamma} \otimes_{\mathbb{Z}} \omega_\Gamma^k)} & \xrightarrow{\mathrm{sh}} & \widetilde{\mathrm{H}}^1(Y_\Gamma, \mathscr{U}_\Gamma^k \otimes_{\mathbb{Z}} \mathbb{C}) \\
\big\downarrow & & \big\downarrow \\
\mathrm{H}^0(Y_\Gamma, \Omega^1_{Y_\Gamma} \otimes_{\mathbb{Z}} \omega_\Gamma^k) \oplus \overline{\mathrm{H}^0(Y_\Gamma, \Omega^1_{Y_\Gamma} \otimes_{\mathbb{Z}} \omega_\Gamma^k)} & \xrightarrow{\mathrm{sh}_\circ} & \mathrm{H}^1(Y_\Gamma, \mathscr{U}_\Gamma^k \otimes_{\mathbb{Z}} \mathbb{C})
\end{array}$$

Lastly, observe that all instances of $\mathrm{Sym}^k_{\mathbb{Z}}(\mathrm{R}^1 f_{\Gamma,*}\mathbb{Z})$ can be replaced with $\mathrm{Sym}^k_{\mathbb{Q}}(\mathrm{R}^1 f_{\Gamma,*}\mathbb{Q})$ and we do so from now onwards.

Definition 3.4. An element of $\mathrm{H}^0(X_\Gamma, \omega_\Gamma^{\otimes k})$ is called a **modular form** of weight k. An element of $\mathrm{H}^0(X_\Gamma, \omega_\Gamma^{\otimes k}(-C))$, where C denotes $X_\Gamma \setminus Y_\Gamma$ viewed as a Weil divisor on X_Γ, is called a **cusp form** of weight k.

Theorem 3.5. *The Kodaira-Spencer map* $\mathrm{KS}_{\mathscr{E}/\mathfrak{h}}$ *is an* $\mathrm{SL}_2(\mathbb{R})$*-equivariant isomorphism.*

Therefore, it descends to an isomorphism $\mathrm{KS}_{\mathscr{E}_\Gamma/Y_\Gamma}\colon \mathscr{E}_\Gamma \to Y_\Gamma$. It then turns out that this extends to the compactification to give an isomorphism $\mathrm{KS}_{\overline{\mathscr{E}_\Gamma}/X_\Gamma}\colon \overline{\mathscr{E}}_\Gamma \to \overline{X}_\Gamma$, and consequently cusp forms can be identified with global sections of $\Omega^1_{X_\Gamma} \otimes \omega_\Gamma^{\otimes(k-2)}$ for $k \geqslant 2$.

4. **Modular curves over $\mathbb{Q}$ and Hecke correspondences**

Define $F_{Y(N)}$ to be the moduli functor

$$\begin{array}{ccc} \mathsf{Sch} & \longrightarrow & \mathsf{Set} \\ S & \mapsto & \left\{ \begin{array}{c} \mathscr{E} \to S \text{ family of elliptic curves,} \\ \text{a pair of sections } P, Q \in \mathscr{E}(S)[N] \text{ such that} \\ (P,Q)\colon (\mathbb{Z}/N\mathbb{Z})_S^{\oplus 2} \to \mathscr{E}[N] \\ \text{is an isomorphism of group } S\text{-schemes.} \end{array} \right\}_{/\cong} \end{array}$$

Theorem 4.1. *For $N \geqslant 3$, $F_{Y(N)}$ is represented by a smooth affine scheme $Y(N)$ of pure relative dimension 1 over* $\operatorname{Spec}\mathbb{Z}[\frac{1}{N}]$.

Define $F_{Y(N;p)}$ to be the moduli functor $\mathsf{Sch} \to \mathsf{Set}$ which associates to a scheme S the isomorphism classes of diagrams

$$\begin{array}{ccccc} E & & \xrightarrow{\ \varphi\ } & & F \\ \uparrow & & & & \uparrow \\ E[N] & & & & F[N] \\ & \overset{\alpha}{\searrow} & & \overset{\alpha'}{\swarrow} & \\ & & (\mathbb{Z}/N\mathbb{Z})_S & & \end{array}$$

where $(E,\alpha),(F,\alpha') \in F_{Y(N)}(S)$ and φ is a p-isogeny, i.e.,

- φ a surjective map of commutative group S-schemes,
- the effective Cartier divisor $E \times_F S \hookrightarrow E$ is of the form $s_1 + s_2 + \cdots + s_p$ where s_i are sections of $E \to S$.

Theorem 4.2. *For $N \geqslant 5$, $F_{Y(N;p)}$ is represented by an affine curve $Y(N;p)$ over* $\operatorname{Spec}\mathbb{Z}[\frac{1}{N}]$.

The following are the universal diagrams:

$$\begin{array}{c} \mathscr{E}(N) \\ \downarrow{\scriptstyle f_N} \\ Y(N) \\ \downarrow \\ \operatorname{Spec}\mathbb{Z}[\frac{1}{N}] \end{array} \qquad \begin{array}{ccccc} \mathscr{E}_1(N) & & \xrightarrow{\ \varphi\ } & & \mathscr{E}_2(N) \\ & \searrow & & \swarrow & \\ & & Y(N;p) & & \\ & & \downarrow & & \\ & & \operatorname{Spec}\mathbb{Z}[\frac{1}{N}] & & \end{array}$$

There are finite étale maps $q_1, q_2\colon Y(N;p) \to Y(N)$ given by sending $((E,\alpha),(F,\alpha'),\cdots) \mapsto (E,\alpha)$ and $((E,\alpha),(F,\alpha'),\cdots) \mapsto (F,\alpha')$, respectively. One way to understand this is to note that the maps $X(N;p)^{\mathrm{an}} \to X(N)^{\mathrm{an}}$, are finite covering maps of compact Riemann surfaces, and use GAGA. Here, $\mathscr{E}_i(N) = \mathscr{E}(N) \times_{Y(N),q_i} Y(N;p)$ for $i = 1,2$. For primes $p \nmid N$, one can informally define the Hecke correspondence T_p as "$q_{1,*}\varphi^* q_2^*$". This definition has the advantage of being "over $\mathbb{Q}$". There is a map $(E,\alpha) \mapsto (E, p^{-1}\alpha)$ which defines an automorphism $I_p\colon Y(N) \to Y(N)$. The diamond operator $\langle p \rangle$ can be defined as "I_p^*".

5. **Hecke action on cohomology**

Define r_1 and r_2 so that the following diagrams are fibered

$$\begin{array}{ccc} \mathscr{E}_1(N) & \longrightarrow & \mathscr{E}(N) \\ {\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle f_N} \\ Y(N;p) & \xrightarrow[q_1]{} & Y(N) \end{array} \qquad \begin{array}{ccc} \mathscr{E}_2(N) & \longrightarrow & \mathscr{E}(N) \\ {\scriptstyle r_2}\downarrow & & \downarrow{\scriptstyle f_N} \\ Y(N;p) & \xrightarrow[q_2]{} & Y(N) \end{array}$$

The action of T_p on $\widetilde{\mathrm{H}}^1(Y(N)^{\mathrm{an}}, \mathrm{Sym}^k_{\mathbb{Q}}(\mathrm{R}^1 f_{N,*}\mathbb{Q}))$ is the composite in the following diagram:

$$\begin{array}{ccccc} \widetilde{\mathrm{H}}^1(Y(N)^{\mathrm{an}}, \mathrm{Sym}^k_{\mathbb{Q}}(\mathrm{R}^1 f_{N,*}\mathbb{Q})) & \xrightarrow{q_2^*} & \widetilde{\mathrm{H}}^1(Y(N;p)^{\mathrm{an}}, \mathrm{Sym}^k_{\mathbb{Q}}(\mathrm{R}^1 r_{2,*}\mathbb{Q})) & \xrightarrow{\varphi^*} & \widetilde{\mathrm{H}}^1(Y(N;p)^{\mathrm{an}}, \mathrm{Sym}^k_{\mathbb{Q}}(\mathrm{R}^1 r_{1,*}\mathbb{Q})) \\ & & & & \| \\ & & \widetilde{\mathrm{H}}^1(Y(N)^{\mathrm{an}}, \mathrm{Sym}^k_{\mathbb{Q}}(\mathrm{R}^1 f_{N,*}\mathbb{Q})) & \xleftarrow[q_{1,*}]{} & \widetilde{\mathrm{H}}^1(Y(N;p)^{\mathrm{an}}, q_1\,\mathrm{Sym}^k_{\mathbb{Q}}(\mathrm{R}^1 r_{2.*}\mathbb{Q})) \end{array}$$

Here we are implicitly using the natural isomorphisms $q_2^*\mathrm{R}^1 f_{N,*}\mathbb{Q} \cong \mathrm{R}^1 r_{2,*}\mathbb{Q}$ and $q_1^*\mathrm{R}^1 f_{N,*}\mathbb{Q} \cong \mathrm{R}^1 r_{1,*}\mathbb{Q}$ given by the theorem of (topological) proper base change. Also, φ^* is induced from natural map $r_{2,*}\mathbb{Q} \to r_{1,*}\mathbb{Q}$ corresponding to $\mathbb{Q} \to \varphi_*\mathbb{Q}$. Lastly, it is clear that diamond operators induce automorphisms of cohomology spaces.

Remark 5.1. One naturally asks if there is a Hecke action on $\mathrm{H}^1(Y(N)^{\mathrm{an}}, \mathrm{Sym}^k_{\mathbb{Q}}(\mathrm{R}^1 f_{N,*}\mathbb{Q}))$. The obstacle with this is that we cannot use proper base change.

6. **The Galois representation**

To obtain the required Galois representation, we utilize the Galois action on ℓ-adic cohomology groups and relate the same to singular cohomology groups via comparision theorems. Artin comparision theorem tells us that

$$\widetilde{\mathrm{H}}^1_{\text{ét}}(Y(N) \otimes_{\mathbb{Z}[1/N]} \overline{\mathbb{Q}}, \mathrm{Sym}^k_{\mathbb{Q}_\ell}(\mathrm{R}^1_{\text{ét}} f_{N,*}\mathbb{Q}_\ell)) \cong \widetilde{\mathrm{H}}^1(Y(N)^{\mathrm{an}}, \mathrm{Sym}^k_{\mathbb{Q}}(\mathrm{R}^1 f_{N,*}\mathbb{Q})) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell.$$

For brevity, denote the $\mathbb{Q}$-vectorspace $\mathrm{H}^1(Y(N)^{\mathrm{an}}, \mathrm{Sym}^k_{\mathbb{Q}}(\mathrm{R}^1 f_{N,*}\mathbb{Q}))$ by W. By the Eichler-Shimura isomorphism, $W \otimes_{\mathbb{Q}} \mathbb{C}$ is basically the space $\mathcal{S}_{k+2}(N)$, along with conjugates thereof, of all cusp forms of weight $k+2$ and level N, which admits a Hecke action. We just saw that Artin comparison theorem gives a linear $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the ℓ-adic completion $W \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. The Hecke operators are defined over $\mathbb{Q}$, so they act on W and the eigenspace decomposition of $W \otimes_{\mathbb{Q}} \mathbb{C}$ is defined over $\overline{\mathbb{Q}}$. We remark that a simple consequence is that Hecke eigenvalues are algebraic numbers. By the property of *multiplicity one*, it follows that the Hecke eigenspace of $W \otimes_{\mathbb{Q}} \mathbb{C}$ containing f is precisely the two-dimensional space $\mathbb{C}f \oplus \mathbb{C}\overline{f}$. From here, it is easily seen that the Hecke eigenspace containing f in $W \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is $\overline{\mathbb{Q}}f \oplus \overline{\mathbb{Q}}\overline{f}$. We also have a Hecke action on $W \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ which commutes with the Galois action. This means that the Hecke eigenspaces of $W \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell$ are Galois invariant. Thus, we obtain a Galois representation

$$\rho_{f,\ell}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell).$$

7. Étale cohomology

Let $a\colon Y(N) \to \operatorname{Spec}\mathbb{Z}[1/N]$ be the structure map. Consider the ℓ-adic sheaf $\mathscr{W} = \widetilde{\mathrm{R}}^1_{\text{ét}} a_*(\operatorname{Sym}^k \mathrm{R}^1_{\text{ét}} f_{N,*}\mathbb{Q}_\ell)$ on $\operatorname{Spec}\mathbb{Z}[1/N]$. Note that f_N is proper and smooth. Therefore, $\mathscr{W}$ is a lisse ℓ-adic sheaf. Let p be a prime not dividing ℓN. By (ind-)smooth base change [Con, Theorem 1.3.5.2], W is the étale stalk of $\mathscr{W}$ at the generic point $\operatorname{Spec}\mathbb{Q} \hookrightarrow \operatorname{Spec}\mathbb{Z}[1/N]$ and $W_p := \widetilde{\mathrm{H}}^1_{\text{ét}}(Y(N) \otimes_{\mathbb{Z}[1/N]} \overline{\mathbb{F}}_p, \operatorname{Sym}^k_{\mathbb{Q}_\ell}(\mathrm{R}^1_{\text{ét}} f_{N,*}\mathbb{Q}_\ell))$ is the étale stalk of $\mathscr{W}$ at $\operatorname{Spec}\mathbb{F}_p \hookrightarrow \operatorname{Spec}\mathbb{Z}[1/N]$. Since $\mathscr{W}$ is lisse, we have an isomorphism of $\mathbb{Q}_\ell$-vectorspaces $W \cong W_p$. Because of various functorialities, this isomorphism is both Hecke and Galois equivariant through a map $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ depending on the choice of geometric point $\operatorname{Spec}\overline{\mathbb{F}}_p \to \operatorname{Spec}\mathbb{Z}[1/N]$. Thus, $\rho_{f,\ell}$ is unramified away from ℓN. An alternative perspective is to look at $\mathscr{W}|_{\operatorname{Spec}\mathbb{Z}[1/\ell N]}$ and consider the monodromy action of $\pi_1^{\text{ét}}(\operatorname{Spec}\mathbb{Z}[1/\ell N])$ on the generic stalk W.

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