The LCM of polynomial sequences at prime arguments

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A word on notations

- We write $a(x) = \mathcal{O}(b(x))$ if there exists an absolute constant c such that $|a(x)| < c \cdot b(x)$ for all sufficiently large x. If $\lim_{x\to\infty} a(x)/b(x) = 0$ then we write $a(x) = o(b(x))$.
- $a(x) \ll b(x)$ means $a(x) < C \cdot b(x)$ for some positive constant C and for all sufficiently large x.
- $a(x) \gg b(x)$ means $a(x) > C \cdot b(x)$ for some positive constant C and for all sufficiently large x.
- We say that $a(x) \sim b(x)$ if $\lim_{x\to\infty} \frac{a(x)}{b(x)} = 1$.
- Throughout the article, p and q will denote primes, and we fix a monic irreducible polynomial $f \in \mathbb{Z}[x]$ of degree $d \geqslant 1$.
- We will often suppress the dependence of constants on f.
- Define $\pi(x)$ to be the number of primes $p < x$ and $\pi(x; m, a)$ to be the number of primes $p < x$ such that $p \equiv a \pmod{m}$.
- For convenience, set $x_{\mathfrak{b}} = x^{1/2}(\log x)^{-B}$.

Introduction

- The Prime Number Theorem is equivalent to

$$\log \operatorname{lcm}\{1, 2, \ldots, n\} \sim n.$$

- Indeed,

$$\log \operatorname{lcm}\{1, 2, \ldots, n\} = \sum_{p \leqslant n} \left\lfloor \frac{\log n}{\log p} \right\rfloor \log p \approx \sum_{p \leqslant n} \frac{\log n}{\log p} \cdot \log p = \pi(n) \log n.$$

- Motivated by this, people investigated $\operatorname{lcm}\{f(1), f(2), \ldots, f(n)\}$ for some irreducible polynomial f.
- However, the growth is not the same for $\deg f \geqslant 2$. It is conjectured that $\log \operatorname{lcm}\{f(1), f(2), \ldots, f(n)\} \sim (d-1)x \log x$ for irreducible polynomials f of degree $d \geqslant 2$.
- We study the analogous problem at prime arguments. That is, $\operatorname{lcm}\{f(p) \mid p < x\}$ for an arbitrary polynomial $f \in \mathbb{Z}[x]$. For simplicity, we will only consider irreducible polynomials f.

Results

Theorem 1 (N. & Jha)

Let $f \in \mathbb{Z}[x]$ be an irreducible polynomial of degree d. Then,

$$\log \operatorname{lcm}\{f(p) \mid p < x\} \gg x^{1-\varepsilon(d)},$$

where $\varepsilon(1) = 0.3735$, $\varepsilon(2) = 0.153$ and $\varepsilon(d) = \exp\left(\frac{-d-0.9788}{2}\right)$ for $d \geqslant 3$.

We remark that $\log \operatorname{lcm}\{f(p) \mid p < x\} \leqslant (d + o(1))x \ll x$ follows from the Prime Number Theorem.

Theorem 2 (N. & Jha)

Let $f \in \mathbb{Z}[x]$ be an irreducible polynomial of degree d. Then, there is a positive proportion of primes p such that $f(p)$ has a prime divisor greater than $p^{1-\varepsilon(d)}$, where $\varepsilon(1) = 0.3735$, $\varepsilon(2) = 0.153$ and $\varepsilon(d) = \exp\left(\frac{-d-0.9788}{2}\right)$ for $d \geqslant 3$.

Setup

- We study the product defined by

$$Q(x) = \prod_{q<x} |f(q)| = \prod_{p} p^{\alpha_p(x)}$$

- Idea is to exploit the fact that the contribution of prime factors less than x^δ is negligible compared to that of prime factors greater than x^δ, where δ is a parameter in $(\frac{1}{2}, 1)$ to be chosen later. Throughout, B will denote some large enough constant.
- Define $\mathrm{res}(m)$ to be the set of residues modulo m which satisfy the congruence $f(x) \equiv 0 \pmod{m}$ and $\mathrm{res}_{\mathrm{num}}(m)$ to be the cardinality of $\mathrm{res}(m)$.
- Note that we have $\mathrm{res}_{\mathrm{num}}(p) \leqslant d$ by Lagrange's theorem and that if $p \nmid \operatorname{disc} f$ then $\mathrm{res}_{\mathrm{num}}(p) = \mathrm{res}_{\mathrm{num}}(p^n)$ for all $n \geqslant 2$ by Hensel's lemma.
- Also define $\sigma(m)$ to be the sum

$$\sum_{r \in \mathrm{res}(m)} \pi(x; m, r),$$

the number of elements in $\{f(p) \mid p < x\}$ divisible by m.

Estimate for $\alpha_p(x)$

Lemma 3 ("α bound")

Let p be a prime. If $p \nmid \operatorname{disc} f$, then

$$\alpha_p(x) = \sum_{p^n < x_{\mathfrak{b}}} \sigma(p^n) + \mathcal{O}\left(\frac{x}{\max\{p, x_{\mathfrak{b}}\} \log x} + \frac{(\log x)^{2B}}{\log p}\right);$$

else if $p \mid \operatorname{disc} f$, we have

$$\alpha_p(x) = \sigma(p).$$

- We only consider the case $p \nmid \operatorname{disc} f$.
- We have
$$\alpha_p(x) = \sum_{n=1}^{\infty} \sigma(p^n) = \sum_{p^n < x} \sigma(p^n) + \sum_{x \leqslant p^n} \sigma(p^n).$$
- When $p^n \geqslant x$, we see that $\sigma(p^n) \leqslant \operatorname{res}_{\text{num}}(p^n) \leqslant d$.
- If p^n divides $f(k)$ for some $1 \leqslant k \leqslant x$, we have $p^n \leqslant f(k) \leqslant f(x) < x^{d+1}$, which implies that $n < (d+1)\frac{\log x}{\log p} \ll \log x / \log p$.

Estimate for $\alpha_p(x)$

- Thus,

$$\alpha_p(x) = \sum_{n=1}^{\infty} \sigma(p^n) = \sum_{p^n < x} \sigma(p^n) + \mathcal{O}\left(\frac{\log x}{\log p}\right).$$

- We split the summation into three intervals:

$$p^n \in [1, x_{\mathfrak{b}}] \cup (x_{\mathfrak{b}}, x^{0.9}] \cup (x^{0.9}, x).$$

- The third summation is small. By routine calculations, it can be shown to be at most $x^{0.2}$.
- The second summation is

$$\sum_{p^n \in (x_{\mathfrak{b}}, x^{0.9}]} \sigma(p^n) = \sum_{p^n \in (x_{\mathfrak{b}}, x^{0.9}]} \sum_{r \in \mathrm{res}(m)} \pi(x; m, r)$$
$$< \sum_{p^n \in (x_{\mathfrak{b}}, x^{0.9}]} \mathrm{res}_{\mathrm{num}}(m) \max_{r \in \mathrm{res}(m)} \pi(x; m, r)$$

Estimate for $\alpha_p(x)$

$$\sum_{p^n \in (x_{\mathfrak{b}}, x^{0.9}]} \sigma(p^n) < \sum_{p^n \in (x_{\mathfrak{b}}, x^{0.9}]} \mathrm{res}_{\mathrm{num}}(m) \max_{r \in \mathrm{res}(m)} \pi(x; m, r)$$

Lemma 4 (Weak Brun-Titchmarsh)

Let $\varepsilon > 0$ be a constant. Then, $\pi(x; m, a) \ll_\varepsilon \frac{x}{\phi(m) \log x}$ for all positive integers $m < x^{1-\varepsilon}$.

Using the above bound, the proof can be completed.

$$\sum_{p^n \in (x_{\mathfrak{b}}, x^{0.9}]} \sigma(p^n) \ll \frac{x}{\max\{p, x_{\mathfrak{b}}\} \log x} + \frac{(\log x)^{2B}}{\log p}.$$

Estimate for small primes

- We define
$$Q_S(x) = \prod_{p < x_{\mathfrak{b}}} p^{\alpha_p(x)},$$
the part of $Q(x)$ consisting of small prime divisors.
- Using "α bound",
$$\begin{aligned}\log Q_S(x) &= \sum_{p<x_{\mathfrak{b}}} \alpha_p(x) \log p \\ &= \sum_{p<x_{\mathfrak{b}}} \left(\sum_{p^n < x_{\mathfrak{b}}} \sigma(p^n) + \mathcal{O}\left(\frac{x}{x_{\mathfrak{b}} \log x} + \frac{(\log x)^{2B}}{\log p} \right) \right) \log p \\ &= \sum_{m<x_{\mathfrak{b}}} \sigma(m)\Lambda(m) + \mathcal{O}\left(\frac{x}{\log x} \right).\end{aligned}$$
- Λ is the **von Mangoldt function** defined as
$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Estimate for small primes

Theorem 5 (Bombieri-Vinogradov)

Let $B \geqslant 6$ *and* $Q \leqslant x^{\frac{1}{2}}(\log x)^{-B}$. *Then,*

$$\sum_{q \leqslant Q} \max_{2 \leqslant y \leqslant x} \max_{(a,q)=1} \left| \pi(y; q, a) - \frac{y}{\phi(q) \log y} \right| \ll_B \frac{x}{(\log x)^{B-5}}.$$

$$\begin{aligned}
\sum_{m < x_{\mathfrak{b}}} \sigma(m)\Lambda(m) &= \sum_{m < x_{\mathfrak{b}}} \sum_{r \in \mathrm{res}(m)} \pi(x; m, r)\Lambda(m) \\
&< \sum_{m < x_{\mathfrak{b}}} \mathrm{res}_{\mathrm{num}}(m) \max_{r \in \mathrm{res}(m)} \pi(x; m, r)\Lambda(m) \\
&\ll \frac{x}{\log x} \sum_{m < x_{\mathfrak{b}}} \frac{\mathrm{res}_{\mathrm{num}}(m)\Lambda(m)}{\phi(m)} + \mathcal{O}\left(\frac{x}{(\log x)^{B-5}}\right) \\
&\vdots
\end{aligned}$$

Estimate for small primes

Lemma 6 (Corollary of §3.3.3.5 of Serre's "Lectures on $N_X(p)$")

Let f be an irreducible integer polynomial and $\mathrm{res}_{num}(m)$ be the number of roots of the congruence $f(x) \equiv 0 \pmod{m}$. Then,

$$\sum_{p<x} \frac{\mathrm{res}_{num}(p)\log p}{p-1} = \log x + R + o(1)$$

for some constant R.

Through some calculation, one finds that

$$\sum_{m<x_{\mathfrak{b}}} \sigma(m)\Lambda(m) \ll \frac{x}{2} - \frac{Bx\log\log x}{\log x} + \mathcal{O}\left(\frac{x}{\log x}\right).$$

Proposition 1

$\log Q_S(x) \ll \frac{x}{2} - \frac{Bx\log\log x}{\log x} + \mathcal{O}\left(\frac{x}{\log x}\right).$

Removing medium-sized primes

- Define the product
$$Q_M(x) = \prod_{x_{\mathfrak{b}} \leqslant p \leqslant x^{1/2}} p^{\alpha_p(x)},$$
the part of $Q(x)$ consisting of medium-sized primes. The main result of this section is the following.

Proposition 2

$\log Q_M(x) \ll \frac{x \log \log x}{\log x}$.

- This means we can just remove medium-sized primes from $\log Q(x)$ and only lose a sublinear quantity.

Removing medium-sized primes

From "α bound", it follows that

$$\begin{aligned}\log Q_M(x) &= \sum_{x_{\mathfrak{b}} \leqslant p \leqslant x^{1/2}} \alpha_p(x) \log p \\ &\ll \sum_{x_{\mathfrak{b}} \leqslant p \leqslant x^{1/2}} \left(\frac{x}{p \log x} + \frac{(\log x)^{2B}}{\log p} \right) \log p \\ &= \frac{x}{\log x} \sum_{x_{\mathfrak{b}} \leqslant p \leqslant x^{1/2}} \frac{\log p}{p} + \mathcal{O}(x^{1/2} (\log x)^{2B}) \\ &\ll \frac{x \log \log x}{\log x}, \qquad \text{(Mertens' theorem)}\end{aligned}$$

as desired.

Theorem 7 (Mertens)

$\sum_{p<x} \log p/p = \log x + O(1)$.

Bounding large primes

Define the product

$$Q_L(x) = \prod_{x^{1/2} < p < x^\delta} p^{\alpha_p(x)},$$

the part of $Q(x)$ consisting of large primes. One carries out a similar analysis and obtains

Proposition 3

$\log Q_L(x) \leqslant (1 + o(1))x \int_{1/2}^{\delta} C(\theta)\, \mathrm{d}\theta.$

where $C(\theta)$ is as in the following theorem.

Theorem 8 (Brun-Titchmarsh, Iwaniec)

Let $\theta = \frac{\log m}{\log x}$. *Then,*

$$\pi(x; m, a) < (C(\theta) + o(1)) \cdot \frac{x}{\phi(m) \log x}$$

for $(C(\theta) = \frac{2}{1-\theta}, \theta \in (0, 1))$ *and* $(C(\theta) = \frac{8}{6-7\theta}, \theta \in [9/10, 2/3])$.

Summary of estimates

- $$\log Q_S(x) = \frac{x}{2} - \frac{Bx \log\log x}{\log x} + \mathcal{O}\left(\frac{x}{\log x}\right)$$
- $$\log Q_M(x) \ll \frac{x \log\log x}{\log x}.$$
- $$\log Q_L(x) \leqslant (1 + o(1))x \int_{1/2}^{\delta} C(\theta)\ \mathrm{d}\theta$$

.

The main bound

- Since $f(x) \sim x^d \implies \log f(x) = d \log x + \mathcal{O}(1)$, it is easy to see that

$$\log Q(x) = \sum_{p<x} (d \log p + \mathcal{O}(1)) = dx + \mathcal{O}(x/\log x).$$

- Define

$$Q_{VL}(x) = \prod_{p \geqslant x^\delta} p^{\alpha_p(x)},$$

the part of $Q(x)$ consisting of primes at least x^δ (*very large* primes).

- $$\log Q_{VL}(x) = \log \frac{Q(x)}{Q_S(x) Q_M(x) Q_L(x)} \geqslant \left(d - \frac{1}{2} - \int_{1/2}^{\delta} C(\theta)\, \mathrm{d}\theta + o(1) \right) x.$$

Finishing the argument

- Define $L(x) = \operatorname{lcm}\{f(p) \mid p < x\}$. Let p be a prime such that $p \geqslant x^{\delta}$. One can check that the exponent of p in $Q(x)$ is at most $\mathcal{O}(x^{1-\delta})$. Indeed, since $p^2 > x$, it follows that p^2 can divide at most one of $f(q)$'s. So most of the exponent comes from p dividing $f(q)$ only once.
- Therefore,

$$\left(d - \frac{1}{2} - \int_{1/2}^{\delta} C(\theta)\,\mathrm{d}\theta + o(1)\right) x \leqslant \log Q_{VL}(x) \ll x^{1-\delta} \sum_{\substack{p \geqslant x^{\delta} \\ p \mid Q(x)}} \log p.$$

- Thus,

$$\log L(x) > \sum_{\substack{p \geqslant x^{\delta} \\ p \mid Q(x)}} \log p \gg x^{\delta}$$

  holds for each δ satisfying $d - \frac{1}{2} - \int_{1/2}^{\delta} C(\theta)\,\mathrm{d}\theta > 0$.
- By routine optimization, it can be obtained that $\delta = 1 - \varepsilon(d)$ works for $\varepsilon(1) = 0.3735$, $\varepsilon(2) = 0.153$ and $\varepsilon(d) = \exp\left(\frac{-d-0.9788}{2}\right)$ for $d \geqslant 3$.

Greatest prime divisor of $f(p)$

- Set $\delta = 1 - \varepsilon(d)$. We saw that

$$\log Q_{VL}(x) = \sum_{q<x} \sum_{\substack{p>x^{\delta} \\ p|f(q)}} \log p \gg x.$$

- Let the number of primes p less than x such that $f(p)$ has a prime divisor greater than x^{δ} be $N(x)$. Note that if $p \mid Q(x)$, then $p < x^{d+1}$ for all large x.
- Thus,

$$N(x) \gg \sum_{q<x} \sum_{\substack{p>x^{\delta} \\ p|f(q)}} 1 \gg \sum_{q<x} \sum_{\substack{p>x^{\delta} \\ p|f(q)}} \frac{\log p}{\log x} \gg \frac{1}{\log x} \sum_{q<x} \sum_{\substack{p>x^{\delta} \\ p|f(q)}} \log p \gg \frac{x}{\log x},$$

which completes the proof.

References

On the Least Common Multiple of Polynomial Sequences at Prime Arguments (with Abhishek Jha), International Journal Of Number Theory (2021)