ALMOST PURITY

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Abstract. We record a proof of the equivalence between the étale site of a perfectoid space and of its tilt.

1. Main theorem

Fix a perfectoid field K with pseudouniformizer π so that $t := \pi^\flat$ is a pseudouniformizer for $K^\flat$. All almost mathematics is performed with respect to $K^{\circ\circ}$ as usual.

Theorem 1.1 (Almost purity)**.** *Let R be a perfectoid K-algebra. Then inverting π gives an equivalence of categories $R^\circ_{\text{afét}} \simeq R_{\text{fét}}$.*

Note that apriori it is not clear if (almost) finite étale algebras over an perfectoid (almost) algebra is perfectoid.

Lemma 1.2 (Perfectoidness passes through finite étale maps)**.**

(a) Let K be of characteristic p. Let R be a perfectoid K-algebra. Let S be a finite étale R-algebra. Then S is uniquely K-perfectoid.

(b) Let R be a perfectoid almost K°-algebra. Let S be an almost finite étale R-algebra. Then S is uniquely almost K°-perfectoid.

Proof.

(a) Let $R \to S$ be finite étale. Let S_0 be the integral closure of R° in S. Topologize S using the pair (S_0, π). Since $S_0^a = S^{\circ a}$, it follows that S° is bounded. Observe that this topology is the same as the one endowed by viewing S as a locally free R-module. Therefore, S° is π-adically complete. We need to verify that S° is semiperfect. But this is trivial because S is perfect. Indeed, perfectness passes through finite étale maps– see [Stacks, Tag 0F6W].

(b) Analogous. See [Bha, Page 49]. □

By Lemma 1.2 and from the main theorem of tilting correspondence of perfectoid algebras, "inverting π" in the context of Theorem 1.1 factors through a subcategory of the R-comma category of Perf_K. This immediately implies that it is fully faithful. The hard part is to prove essential surjectivity, which is the object of Section 3.

Corollary 1.3 (Tilting invariance of étale site)**.** *Let R be a perfectoid K-algebra. Then tilting and untilting induces the following chain of equivalences*

$$R^\flat_{\text{fét}} \simeq R^{\flat\circ}_{\text{afét}} \simeq (R^{\flat\circ}/t)_{\text{afét}} \simeq (R^\circ/\pi)_{\text{afét}} \simeq R^\circ_{\text{afét}} \simeq R_{\text{fét}}.$$

Proof. Almost purity (Theorem 1.1) for characteristic p perfectoid algebras, which will be proven in Section 2, is going to give us the first equivalence. The functor in the second equivalence is "mod t reduction". By Lemma 1.2, the data of an object of $R^{\flat\circ}_{\text{afét}}$ is equivalent to the data of an inverse system of almost finite étale $R^{\flat\circ}/t^n$-algebras indexed by n. An almost analog of topological invariance of étale site then gives us that the second functor is an equivalence. The fourth functor is an equivalence due to the

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same reason. The third functor is identity. The last functor $A \mapsto A_*[\frac{1}{\pi}]$ is an equivalence by Theorem 1.1. □

Proposition 1.4 (Almost purity for perfectoid fields)**.** *Theorem 1.1 is true for $R = K$.*

Proof. See [Sch12, Page 283]. □

2. **Almost purity in characteristic p**

The goal of this section is to prove Theorem 1.1 when K has characteristic p. We will show that if R is a perfectoid K-algebra and S is finite étale over R, which has to be uniquely perfectoid by Lemma 1.2, then S° is almost finite étale over R°. It is harmless to assume $\operatorname{Spec} R$ is connected so that $\operatorname{Spec} S \to \operatorname{Spec} R$ is a finite étale *cover*. We first handle the Galois case, that is, assume that $\operatorname{Spec} S \to \operatorname{Spec} R$ is a finite étale Galois cover. By how the perfectoid structure is endowed on S, we know that $S_0 \hookrightarrow S^\circ$ is an almost isomorphism, where S_0 is the integral closure of R° in S. Therefore, it suffices to show that S_0 is almost finite étale over R. From now on, replace S° with S_0 (so S° is not the subring of all powerbounded elements anymore; this is purely a cosmetic change of notation). Denote $\eta\colon R^\circ \to S^\circ$. Since $\eta[\frac{1}{t}]$ is finite étale, there is a diagonal idempotent $e \in S \otimes_R S$ such that $t^n e \in S^\circ \otimes_{R^\circ} S^\circ$ for some n. By taking pth roots, we get $t^c e \in S^\circ \otimes_{R^\circ} S^\circ$ for arbitrarily small $c > 0$. This implies that e is an almost element which works as a diagonal idempotent in the almost category. Hence, η is almost unramified. Fix some $\varepsilon \in K^{\circ\circ}$. Write $\varepsilon e = \sum_{i=1}^n a_i \otimes b_i \in S^\circ \otimes_{R^\circ} S^\circ$. Consider

$$f\colon S^\circ \to R^{\circ n}, \quad s \mapsto (\operatorname{Tr}_{S/R}(sb_1), \dots, \operatorname{Tr}_{S/R}(sb_n)),$$
$$g\colon R^{\circ n} \to S^\circ, \quad (r_1, \dots, r_n) \mapsto \sum_{i=1}^n a_i r_i.$$

To say that f is well-defined, we need to verify $\operatorname{Tr}_{S/R}(S^\circ) \subset R^\circ$. This is clear because the characteristic polynomial of the multiplication by $\alpha \in S^\circ$ map is equal to $\prod_{\sigma \in \operatorname{Gal}(S/R)}(X - \sigma(\alpha))$ and hence $\operatorname{Tr}_{S/R}(\alpha) = \sum_{\sigma \in \operatorname{Gal}(S/R)} \sigma(\alpha)$, which is clearly an element of R°. Then one verifies that $g \circ f$ is given by multplication by ε. To summarise, the multiplication by t^{1/p^n} map on S° factors as a direct summand of $R^{\circ r_n}$ for some r_n for each n. One then deduces that the relevant Ext-groups are almost vanishing and hence S° is almost projective. In fact, we claim that $R^\circ \to S^\circ$ is an almost finite étale *cover*, i.e., it is almost faithfully flat. We prove something stronger– $R^\circ \to S^\circ$ is almost split. It is enough to show that $K^{\circ\circ} R^\circ \subset \operatorname{Tr}_{S/R} S^\circ$. For if $\operatorname{Tr}_{S/R}(s) = t^{1/p^n}$ say, then we can consider $S^\circ \to R^\circ$ given by $x \mapsto \operatorname{Tr}_{S/R}(sx)$ which when restricted to R° acts as multiplication by t^{1/p^n}. Since $R \to S$ is finite étale, it follows that $\operatorname{Tr}_{S/R}(S) = R$. This assertion can be checked after a faithfully flat base change following which it is immediate. So pick some $f \in S^\circ$ so that $\operatorname{Tr}_{S/R}(f) = t^c$ for some $c \geqslant 0$. It is easily verified that the trace map commutes with Frobenius. Therefore, $\operatorname{Tr}_{S/R}(f^{1/p^n}) = t^{c/p^n}$ for $n \geqslant 0$, which proves the desired inclusion.

For the general case, choose a finite étale cover $\operatorname{Spec} L \to \operatorname{Spec} S$ so that $\operatorname{Spec} L \to \operatorname{Spec} R$ and $\operatorname{Spec} L \to \operatorname{Spec} S$ are finite Galois. Such an L exists by the theory of étale fundamental groups. Note that L is also perfectoid. From our previous work, we know that $S^\circ \to L^\circ$ and $R^\circ \to L^\circ$ are almost finite étale and almost faithfully flat. The assertion that $R^\circ \to S^\circ$ is almost finite étale can be checked after base changing via $R^\circ \to L^\circ$ by almost faithfully flat descent. Indeed, the formalism of flat descent carries over to the almost category by virtue of the adjoint $A \mapsto A_{!!}$, which preserves faithful flatness. Therefore, it suffices to show that $L^\circ \to S^\circ \otimes_{R^\circ} L^\circ$ is almost finite étale. By yoga of perfectness, one can show that $S^\circ \widehat{\otimes}_{R^\circ} L^\circ$ is almost isomorphic to $(S \otimes_R L)^\circ$. However, $S \otimes_R L \simeq S \otimes_R S \otimes_S L \simeq (S \otimes_R S) \otimes_S L$, which is isomorphic to an algebra of the form $L \times L \times \cdots \times L$ since $R \to S$ is finite étale. We conclude that $L^\circ \to S^\circ \otimes_{R^\circ} L^\circ$ is almost finite étale after t-adic completion. In particular, this implies that $L^\circ \to S^\circ \otimes_{R^\circ} L^\circ$ is almost finite étale mod t. This completes the proof because $L^\circ_{\text{afét}} \simeq (L^\circ/t)_{\text{afét}}$ by an almost analog of topological invariance of étale site.

3. **Almost purity in characteristic** 0

Definition 3.1. (1) A map $(A, A^+) \to (B, B^+)$ of affinoid Tate rings is **finite étale** if $A \to B$ is finite étale and B^+ is the integral closure of A^+ in B.

(2) A map $f : X \to Y$ of adic spaces is **finite étale** if there exists an open affinoid cover $\{V_i\}$ of Y such that $U_i = f^{-1}(V_i)$ is affinoid, and $(\mathscr{O}_Y(V_i), \mathscr{O}_Y^+(V_i)) \to (\mathscr{O}_X(U_i), \mathscr{O}_X^+(U_i))$ is finite étale.

(3) A map $(A, A^+) \to (B, B^+)$ of affinoid K-algebras is **strongly finite étale** if it is finite étale and $A^+ \to B^+$ is almost finite étale.

(4) A map $f : X \to Y$ of (perfectoid) adic spaces is **strongly finite étale** if there exists an open affinoid (perfectoid) cover $\{V_i\}$ of Y such that $U_i = f^{-1}(V_i)$ is affinoid (perfectoid), and $(\mathscr{O}_Y(V_i), \mathscr{O}_Y^+(V_i)) \to (\mathscr{O}_X(U_i), \mathscr{O}_X^+(U_i))$ is strongly finite étale.

Remark 3.2. It is not clear apriori if a (strongly) finite étale map of adic spaces which are perfectoid is a (strongly) finite étale map of perfectoid spaces.

For essential surjectivity, it suffices to show that if $(A, A^+) \to (B, B^+)$ is a finite étale map of affinoid K-algebras with A perfectoid then B is also perfectoid and $A^+ \to B^+$ is almost finite étale. Indeed, we may just take $A^+ = A^\circ$ and $B^+ = B^\circ$ to derive Theorem 1.1.

Lemma 3.3. *Strong finite étaleness of perfectoid spaces is preserved under perfectoid base-change.*

Proof. Clear from the construction of tensor products in the category of perfectoid K-algebras. □

Proposition 3.4 ([GR03, Proposition 5.4.53])**.** *Let A be a flat K°-algebra that is Henselian along (π) and $\widehat{A}$ be its completion. Then $A[\frac{1}{\pi}]_{\text{fét}} \simeq \widehat{A}[\frac{1}{\pi}]_{\text{fét}}$.*

Proposition 3.5. *The natural forgetful functor gives an equivalence $(A, A^+)_{\text{fét}} \simeq A_{\text{fét}}$.*

Proof. Clear. □

Corollary 3.6 (Étale site commutes with colimit for complete uniform affinoid K-algebras)**.** *Let (A_i, A_i^+) be a filtered system of complete uniform affinoid K-algebras, and (A, A^+) be their colimit in the category of complete uniform affinoid K-algebras. Then any finite étale A-algebra comes as the base-change of some finite étale A_i-algebra. In other words, $2\text{-}\operatorname{colim}(A_i)_{\text{fét}} \simeq A_{\text{fét}}$.*

Proof. Let us recall how the colimit is constructed. We set A^+ to be the π-adic completion of the ring-theoretic colimit $B^+ := \operatorname{colim} A_i^+$, set $A := A^+[\frac{1}{\pi}]$ and make (A^+, π) a ring of definition. Therefore, by Proposition 3.4, we have $A_{\text{fét}} \simeq B^+[\frac{1}{\pi}]_{\text{fét}} = (\operatorname{colim} A_i)_{\text{fét}}$. It is then a routine exercise to show that étale site commutes with colimit for ordinary rings. □

Proposition 3.7 (Strongly finite étale maps form a stack)**.** *Let $f : X \to Y$ be a strongly finite étale map of perfectoid spaces. Let $V \hookrightarrow Y$ be an affinoid perfectoid open. Then its preimage U is also affinoid perfectoid and the map*

$$(\mathscr{O}_Y(V), \mathscr{O}_Y^+(V)) \to (\mathscr{O}_X(U), \mathscr{O}_X^+(U))$$

is strongly finite étale.

Proof. Routine "Noetherian approximation" method to reduce to the rigid-analytic setting. See [Sch12, Proposition 7.6]. □

Proof of Theorem 1.1. We must show that the fully faithfuly functor $R^{+a}_{\text{afét}} \to R_{\text{fét}}$ is essentially surjective. Let S be a finite étale R-algebra. We must show that S comes from an almost finite étale R^{+a}-algebra. Thanks to Proposition 3.7, we may pass to adic spaces and instead work locally on $X := \operatorname{Spa}(R, R^+)$. Fix

$x \in X$. We want to solve the problem for some neighborhood of x. By formalism of colimits (Corollary 3.6), we expect that we'd be done if we show that there is a strongly finite étale map of the form $\mathrm{Spa}(S\widehat{\otimes}_R\widehat{\kappa(x)}, -) \to \mathrm{Spa}(\widehat{\kappa(x)}, \widehat{\kappa(x)^+})$. Indeed, recall that $\mathrm{colim}_{x\in U}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \simeq (\widehat{\kappa(x)}, \widehat{\kappa(x)^+})$. By almost purity for perfectoid fields (Proposition 1.4), we know that there exist (a *unique*) S_x^+ making $\mathrm{Spa}(S\widehat{\otimes}_R\widehat{\kappa(x)}, S_x^+) \to \mathrm{Spa}(\widehat{\kappa(x)}, \widehat{\kappa(x)^+})$ strongly finite étale. This requires some explanation. Indeed, almost purity for perfectoid fields supplies us an almost finite étale $\widehat{\kappa(x)^+}^a$-algebra, say S_x^{+a}, which we know is almost perfectoid by Lemma 1.2. Now set S_x^+ to be the integral closure of $\widehat{\kappa(x)^+}$ in $S\widehat{\otimes}_R\widehat{\kappa(x)}$ and make $(S\widehat{\otimes}_R\widehat{\kappa(x)}, S_x^+)$ into an affinoid perfectoid $\widehat{\kappa(x)}$-algebra by topologizing in the natural manner. Since $(S_x^{+a})_*$ is also integrally closed, it follows that $\widehat{\kappa(x)^+}$ is a subring of $(S_x^{+a})_*$ both containing $\widehat{\kappa(x)}^{\circ\circ}(S\widehat{\otimes}_R\widehat{\kappa(x)})^\circ$ and contained in $(S\widehat{\otimes}_R\widehat{\kappa(x)})^\circ$. Therefore, S_x^+ is almost isomorphic to $(S_x^{+a})_*$, which shows that S_x^+ is almost finite étale over $\widehat{\kappa(x)^+}$. To 'spread' S_x^+ to an almost finite étale algebra over a neighborhood we'd like to have the following equivalence

$$2\text{-}\mathrm{colim}\,\mathcal{O}_X^+(U)_{\text{afét}} \simeq \widehat{\kappa(x)^+}_{\text{afét}}.$$

By the already proven equivalences in Corollary 1.3, the validity of the above equivalence is compatible with tilting. In characteristic p, the above equivalence holds by Corollary 3.6 and almost purity in characteristic p (Section 2). Therefore, there exists an open affinoid perfectoid neighborhood U of x so that there exists some S_U^{+a} which is almost finite étale over $\mathcal{O}_X^+(U)^a$. Set $S_U = (S_U^{+a})_*[\frac{1}{\pi}]$ and S_U^+ as the integral closure of $\mathcal{O}_X^+(U)$ in S_U. Topologize in such a way that (S_U, S_U^+) is a affinoid perfectoid $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$-algebra. By similar reasoning as earlier, we deduce that $\mathcal{O}_X^+(U) \to S_U^+$ is almost finite étale. Thanks to the following 2-commuting square, we know that S_U agrees with $S\widehat{\otimes}_R\mathcal{O}_X(U)$:

$$\begin{array}{ccc} 2\text{-}\mathrm{colim}\,\mathcal{O}_X^+(U)_{\text{afét}} & \overset{\simeq}{\longleftrightarrow} & \widehat{\kappa(x)^+}_{\text{afét}} \\ \Big\downarrow{\scriptstyle (-)_*[\frac{1}{\pi}]} & & \Big\downarrow{\scriptstyle (-)_*[\frac{1}{\pi}]} \\ 2\text{-}\mathrm{colim}\,\mathcal{O}_X(U)_{\text{fét}} & \overset{\simeq}{\longleftrightarrow} & \widehat{\kappa(x)}_{\text{fét}} \end{array}$$

The left vertical arrow is an equivalence because all the other arrows are. From now on, identify S_U with $S\widehat{\otimes}_R\mathcal{O}_X(U)$. To summarize, we have an open affinoid perfectoid cover $X = \cup_i U_i$ with strongly finite étale maps $\mathrm{Spa}(S_{U_i}, S_{U_i}^+) \to U_i$ where $S_{U_i} = S\widehat{\otimes}_R\mathcal{O}_X(U_i)$. Of course, $S\widehat{\otimes}_R\mathcal{O}_X(U)$ glue to a sheaf $\widetilde{S}$. Indeed, since S is R-flat and completion is exact, the Čech complex of $\mathcal{O}_X$ remains exact after applying $S\widehat{\otimes}_R(-)$. This also means that we can functorially recover S from $\widetilde{S}$ as $\mathrm{H}^0(X, \widetilde{S}) = S$. Since $S_{U_i}^+$ is computed from taking the integral closure of $\mathcal{O}_X^+(U_i)$ in S_{U_i}, it follows from Lemma 3.3 that $S_{U_i}^+$ glue to give a sheaf $\widetilde{S}^+$. Therefore, $\mathrm{Spa}(S_{U_i}, S_{U_i}^+)$ glue to give a perfectoid space Y which is strongly finite étale over X. This also means by Proposition 3.7 that $\widetilde{S}^+$ is given on rational subsets U as the integral closure of $\mathcal{O}_X^+(U)$ in $\widetilde{S}(U)$ and that Y is affinoid perfectoid with $\mathcal{O}_Y(Y) = \mathrm{H}^0(X, \widetilde{S}) = S$, $\mathcal{O}_Y^+(Y)$ the integral closure of R^+ in S, and $\mathcal{O}_Y^+(Y)$ almost finite étale over R^+. □

References

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