Perfectoid Rings and Tilting

Definition 0.1 (Perfectoid Field). A *perfectoid field* is a topological field K complete with respect to a nonarchimedean valuation $|\cdot|$ such that:

(1) The value group $|K^*|$ is not discrete.

(2) The Frobenius $x \mapsto x^p$ on $\mathcal{O}_K/p = K^\circ/p$ is surjective.

We call an element $\varpi \in K^*$ with $|p| \le |\varpi| < 1$ a *pseudo-uniformizer*.

Example.

Any perfect nonarchimedean field K is perfectoid. In particular, $\widehat{\mathbb{F}_p((t))[t^{1/p^\infty}]} = \mathbb{F}_p((t^{1/p^\infty}))$. In fact, any characteristic p perfectoid ring is perfect.

$\widehat{\mathbb{Q}_p(p^{1/p^\infty})} = \mathbb{Q}_p((p^{1/p^\infty}))$ is perfectoid. Indeed, it is Tate ring since $\mathbb{Z}_p[[p^{1/p^\infty}]]$ is a ring of definition and $p^{1/p}$ is a topologically nilpotent unit, moreover $(p^{1/p})^p \mid p$. It is complete by definition and uniform since the ring of power bounded elements $\mathbb{Z}_p((p^{1/p^\infty}))$ is a ring of definition. We have already constructed a pseudo-uniformizer with $\varpi^p \mid p$, so it remains to check that the Frobenius map $\mathbb{Z}_p[[p^{1/p^\infty}]]/p^{1/p} \to \mathbb{Z}_p[[p^{1/p^\infty}]]/p$ is an isomorphism. But this is just the map $\mathbb{F}_p[t^{1/t^\infty}]/t^{1/p} \to \mathbb{F}_p[t^{1/p^\infty}]/t$ which is an isomorphism by perfectness of $\mathbb{F}_p[t^{1/p^\infty}]$.

$\widehat{\mathbb{Q}_p(\mu_{p^\infty})} = \mathbb{Q}_p((\mu_{p^\infty}))$ is perfectoid. The fact that it is a complete uniform Tate ring is the same argument as above using the fact that the ring of integers in $\mathbb{Q}_p(\zeta_{p^n})$ is $\mathbb{Z}_p[\zeta_{p^n}]$. $1-\zeta_p$ is a pseudo-uniformizer with $(1-\zeta_p)^p \mid p$. Finally, it suffices to show surjectivity of Frobenius on $\mathbb{Z}_p[[\mu_{p^\infty}]]/p$. But we have

$$\mathbb{Z}_p[[\mu_{p^\infty}]]/p = \mathbb{Z}_p[\mu_{p^\infty}]/p = (\operatorname{colim} \mathbb{Z}_p[X]/(X^{p^n}-1))/p = \operatorname{colim} \mathbb{F}_p[X]/(X-1)^{p^n} = \mathbb{F}_p[t^{1/p^\infty}]/t$$

and surjectivity of Frobenius on the latter is clear. ◇

Proposition 0.2 (Basic Properties of Perfectoid Fields). Let K be a perfectoid field. Then

1. The value group $|K^*|$ is p-divisible.
2. There exists a pseudo-uniformizer ϖ with a compatible system of p-th power roots. Moreover, $K^{\circ\circ} = (\varpi^{1/p^\infty})$ is the maximal ideal of K°. In particular, $(K^{\circ\circ})^2 = K^{\circ\circ}$ and $K^{\circ\circ}$ is a flat K°-module. Moreover, K° has Krull dimension 1.

Proof. For (1), suppose first that $x \in K^\circ$ with $|p| < |x| \le 1$. By perfectoidness, we can find $y, z \in K^\circ$ such that $y^p = x + pz$. Since $|x| > |p| \ge |pz|$, it follows that $|y|^p = |x|$. Thus $|x|$ is divisible by p. In general, since $\varpi^p \mid p$, we may write $p = \varpi^p x$ with $x \in K^\circ$. In other words, $|p| = |\varpi|^p|x| < |x| \le 1$ using that ϖ is topologically nilpotent so that $|\varpi| < 1$. Thus $|x| = |y|^p$, so that $|p|$ is also p divisible. Finally, if $x \in K^*$, then replacing x with $p^n|x|^{\pm 1}$ for $n \in \mathbb{Z}$, we may assume that $|p| \le |x| \le 1$. But then the result is clear for $|x|$.

For (2), we will prove everything under the assumption that such a ϖ exists. This fact will be deduced later for a much larger class of rings. Since K° is a rank 1 valuation ring, i.e. its value group embeds into $\mathbb{R}_{\ge 0}$, its Krull dimension is 1. Moreover, since ϖ is nonzero, the ring K°/ϖ is a local ring of dimension 0, so its maximal ideal is its nilradical $\operatorname{rad}(K^\circ/\varpi)$. Evidently, $(\varpi^{1/p^\infty}) \subset \operatorname{rad}(K^\circ/\varpi)$ and moreover, the quotient $K^\circ/(\varpi^{1/p^\infty})$ is perfect, hence reduced. Thus $(\varpi^{1/p^\infty}) = \operatorname{rad}(K^\circ/\varpi)$ and the maximal ideal of K° is the pre-image of this ideal, which is also (ϖ^{1/p^∞}). Finally, since $(K^{\circ\circ})^p = (\varpi^{1/p^\infty}) = K^{\circ\circ}$, we find that $(K^{\circ\circ})^2 = K^{\circ\circ}$. For the flatness claim, we note that any torsion-free module over a valuation ring is flat. □

We will now generalize the situation beyond perfectoid fields since many of the basic properties of tilting hold in much greater generality. For this we need some definitions coming from p-adic geometry, namely Huber rings.

Definition 0.3 (Huber and Tate Rings). A topological ring A is a *Huber ring* if there is an open subring A_0 and a finitely generated ideal $I \subset A_0$ such that the subspace topology on $A_0 \subset A$ coincides with the I-adic topology. The subring A_0 is called the *ring of definition* and the ideal I the *ideal of definition*.

Let A be a Huber ring. A subset $S \subset A$ is called *bounded* if for any open subset $U \subset A$ there exists an open subset $V \subset A$ such that $\bigcup_{s \in S} sV \subset U$. An element $f \in A$ is called *power-bounded* if the set $\{f^n : n \in \mathbb{N}\} \subset A$ is bounded. We denote the set of all power-bounded elements by A°. An element $f \in A$ is called *topologically nilpotent* if $f^n \to 0$ as $n \to \infty$, that is to say for any open neighborhood U of 0, there exists an $N \in \mathbb{N}$ such that $\{f^n : n \geq N\} \subset U$. We denote the set of topologically nilpotent elements by $A^{\circ\circ}$.

A Huber ring A is called *Tate* if there exists a topologically nilpotent unit $\varpi \in A$, i.e. an element ϖ of $A^* \cap A^{\circ\circ}$. Any such topologically nilpotent unit is called a *pseudo-uniformizer*.

A Huber ring A is called *uniform* if the set $A^\circ \subset A$ of power bounded elements is itself bounded.

Remark. We will primarily be interested in complete uniform Tate rings. For these, the condition can be described more cleanly. Since any ideal of definition is automatically a subring of A° and any open bounded subring of A° is automatically a ring of definition, the condition that A be uniform is equivalent to asking that A° be a ring of definition. Moreover, the condition that A be Tate is equivalent to A° having the ϖ-adic topology with $A = A^\circ[1/\varpi]$. In other words, a topological ring A is a complete uniform Tate ring if $A = A^\circ[1/\varpi]$ and such that A° is complete with respect to the ϖ-adic topology. In this case, we may also define a submultiplicative norm $|x| = \inf\{2^{-n} : x \in \varpi^n A_\circ, n \in \mathbb{Z}\}$ which induces the topology on A. This makes A a Banach ring. Using this norm we may also rephrase the condition that A is a complete uniform Tate ring one final time as follows. A ring A is a complete uniform Tate ring if $A = B[1/\varpi]$ for a ϖ-complete ϖ-torsion-free totally integrally closed ring B containing ϖ.

Here the *total integral closure* of $R \to S$ is the set of $f \in S$ such that $\{f^n : n \in \mathbb{N}\}$ is contained in a finitely generated R-submodule of S. This agrees with the usual integral closure if R is Noetherian, but not in general. A subring $R \subset S$ is *totally integrally closed* if R equals its total integral closure in S. It is not necessarily true that the total integral closure of R in S is totally integrally closed.

Definition 0.4 (Perfectoid Ring). A *perfectoid ring* is a complete uniform Tate ring A with a pseudo-uniformizer ϖ such that $\varpi^p \mid p$ in A° and the Frobenius map $A^\circ/\varpi \to A^\circ/\varpi^p$ is an isomorphism. We denote the category of perfectoid rings by Perf. A *perfectoid field* is a perfectoid ring which is a nonarchimedean field.

Example.

1. Let A be a perfectoid ring. Then $\widehat{A[T^{1/p^\infty}]} = A\langle T^{1/p^\infty}\rangle$ is perfectoid. It is obviously still a complete uniform Tate ring and the same pseudo-uniformizer ϖ works. It suffices to show that Frobenius is surjective $\mod p$, but this is clear.
2. Let $R = \mathbb{Z}_p[[\mu_{p^\infty}, T^{1/p^\infty}]]\langle (p/T)^{1/p^\infty}\rangle[1/T]$. Then R is perfectoid. It is complete and uniform Tate ring by definition and we may take the pseudo-uniformizer to be $\varpi = T^{1/p}$ since then $\varpi^p = T \mid p$. Finally, the same computation as with the cyclotomic case shows that Frobenius is surjective. This example is interesting since it does not contain a field.

◇

Remark. At this point it is good to make a remark that the uniformity of perfectoid rings implies in particular that any perfectoid ring A is reduced. Indeed, if $x \in A$ is nilpotent, then $Ax \subset A^\circ$ and A° is not bounded since $A = A^\circ[1/\varpi]$ is not bounded. This means in particular that notions such as infinitesimal thickenings or differentials are not useful concepts in the perfectoid world. For example, although one may define Kähler differentials, they will generally not work well. Let us give an explicit example:

Exercise. Let K be a perfectoid field of characteristic 0. Show that $\Omega^1_{K^\circ/\mathbb{Z}_p}$ is nonzero but that the p-adic completion $\widehat{\Omega}^1_{K^\circ/\mathbb{Z}_p}$ is always zero.

Proof. For the first part, general properties of Kähler differentials implies that $\Omega^1_{K^\circ/\mathbb{Z}_p}[1/p] = \Omega^1_{K^\circ[1/p]/\mathbb{Q}_p} = \Omega^1_{K/\mathbb{Q}_p}$. Thus it suffices to show that this is nonzero. Let $\mathbb{Q}_p \subset L \subset K$ be the maximal purely transcendental subfield of $K/\mathbb{Q}_p$. Then K/L is an algebraic extension. i.e. $L \to K$ is a filtered colimit of finite separable (i.e. étale) extensions and $L \to K$ is formally étale. This gives an exact sequence

$$0 \to \Omega^1_{L/\mathbb{Q}_p} \to \Omega^1_{K/\mathbb{Q}_p} \to \Omega^1_{K/L} \to 0$$

and since K/L formally étale implies K/L formally unramified implies that $\Omega^1_{K/L} = 0$, we find that $\Omega^1_{L/\mathbb{Q}_p} \cong \Omega^1_{K/\mathbb{Q}_p}$. Since $L/\mathbb{Q}_p$ is purely transcendental, we may write $L = \mathbb{Q}_p(\{x_i : i \in I\})$ and therefore

$$\Omega^1_{L/\mathbb{Q}_p} = \left(\bigoplus_{i \in I} L dx_i\right)\left[\frac{1}{p}\right]$$

Thus it suffices to show that $|I| > 0$. Indeed, since K is perfectoid, $K/\mathbb{Q}_p$ is not finite, since then K would be discretely valued. Thus $K/\mathbb{Q}_p$ is infinite. Since K is complete, this further implies that $|K| > |\mathbb{Q}_p|$ which implies that (1) $K/\mathbb{Q}_p$ is not algebraic, and (2) that $K/\mathbb{Q}_p$ has uncountable transcendence degree. Thus $|I|$ is uncountable and $\Omega^1_{K^\circ/\mathbb{Z}_p}$ uncountably generated.

For the second part, note that for any $x \in K^\circ$, the surjectivity of Frobenius implies that $x = y^p + pz$ for $y, z \in K^\circ$. Thus $dx = p(dy + dz)$ and $p\Omega^1_{K^\circ/\mathbb{Z}_p} = \Omega^1_{K^\circ/\mathbb{Z}_p}$. This then implies that $\widehat{\Omega}^1_{K^\circ/\mathbb{Z}_p} = 0$. □

Theorem 0.5 (Kedlaya).

1. Any Tate ring A has a topology induced by a submultiplicative norm.
2. Any perfectoid ring K whose underlying ring is a field has a topology induced by a nonarchimedean norm on K. In particular, a perfectoid field is the same as a perfectoid ring whose underlying ring is a field.
3. Any perfectoid ring A containing a nonarchimedean field K has a topology induced by a submultiplicative Banach K-algebra norm. In particular, A perfectoid ring A over a perfectoid field K is the same as a perfectoid K-algebra.

Let A be a perfectoid ring. Since the p-th power map is a map of multiplicative monoids on A, we may define $A^\flat = \lim_{x \mapsto x^p} A$ as a topological multiplicative monoid. The starting point of the theory of perfectoid spaces is noticing that tilting has good properties:

Theorem 0.6 (Basic Properties of Tilting). Let A be a perfectoid ring. Then

1. $A^\flat$ becomes a complete perfect Tate $\mathbb{F}_p$-algebra when addition is defined by

$$(x^{(0)}, x^{(1)}, \ldots) + (y^{(0)}, y^{(1)}, \ldots) = (z^{(0)}, z^{(1)}, \ldots)$$
$$z^{(k)} = \lim_{n \to \infty} (x^{(n+k)} + y^{(n+k)})^{p^n}$$

2. The subset $A^{\flat\circ}$ is given by the topological ring isomorphism

$$A^{\flat\circ} = \lim_{x \mapsto x^p} A^\circ \cong \lim_{\Phi} A^\circ/p \cong \lim_{\Phi} A^\circ/\varpi$$

where Φ denotes Frobenius and $\varpi \mid p$ is a pseudo-uniformizer in A°. Here all inverse limits are given the inverse limit of the discrete topology.

3. There is a pseudo-uniformizer $\varpi \in A^\circ$ such that $\varpi^p \mid p$ in A° that admits a sequence of p-th power roots ϖ^{1/p^n} giving rise to an element $\varpi^p = (\varpi, \varpi^{1/p}, \ldots) \in A^{\flat\circ}$ which is a pseudo-uniformizer of $A^\flat$ making the latter into a perfectoid ring with $A^\flat = A^{\flat\circ}[1/\varpi^\flat]$ and $A^{\flat\circ}/\varpi^\flat \cong A^\circ/\varpi$.

Proof. (Part 1) We first have to show that this is in fact well defined. Indeed, for each $n \geq 0$ and $m' \geq m \geq 0$, we have

$$(x^{(n+m')} + y^{(n+m')})^{p^{m'-m}} = (x^{(n+m')})^{p^{m'-m}} + (y^{(n+m')})^{p^{m'-m}} \equiv x^{(n+m)} + y^{(n+m)} \mod p$$

and hence $(x^{(n+m')} + y^{(n+m')})^{p^{m'}} \equiv (x^{(n+m)} + y^{(n+m)})^{p^m} \mod p^{m+1}$. Hence the sequence $(x^{(n+m)} + y^{(n+m)})^{p^m}$

converges in A. Moreover,

$$(x+y)^{(k)} = \lim_{n\to\infty}(x^{(n+k)}+y^{(n+k)})^{p^n} = \lim_{n\to\infty}(x^{(n+k+1)}+y^{(n+k+1)})^{p^{n+1}} = ((x+y)^{(k+1)})^p$$

Commutativity is clear. For associativity, we note that

$$\begin{aligned}(x+y+z)^{(k)} &= \lim_{n\to\infty}((x+y)^{(n+k)}+z^{(n+k)})^{p^n} = \lim_{n\to\infty}(\lim_{m\to\infty}(x^{(m+n+k)}+y^{(m+n+k)})^{p^m}+z^{(n+k)})^{p^n} \\ &= \lim_{n\to\infty}(x^{(n+k)}+y^{(n+k)}+z^{(n+k)})^{p^n}\end{aligned}$$

Using the fact that $(x^{(m+n+k)}+y^{(m+n+k)})^{p^m} \equiv x^{(n+k)}+y^{(n+k)} \mod p$. To show that additive inverses exist, we simply set $(-x)^{(n)} = -x^{(n)}$ if p is odd and $(-x)^{(n)} = x^{(n)}$ if n is even. In the former case, it is immediate that $-x$ is the desired additive inverse. If $p=2$, then we have

$$(x+-x)^{(k)} = \lim_{n\to\infty}(2x^{(n+k)})^{2^n} = x^{(k)}\lim_{n\to\infty}2^{2^n} = 0$$

From the equality constructed using associativity, we immediately find that $(px)^{(k)} = \lim_{n\to\infty}(px^{(n+k)})^{p^n} = x^{(k)}\lim_{n\to\infty}p^{p^n} = 0$. Thus $A^\flat$ is an $\mathbb{F}_p$-algebra. The fact that $A^\flat$ is complete is immediate as it has the inverse limit topology from a sequence of complete spaces. The fact that $A^\flat$ is Tate will then follow from the construction of the pseudo-uniformizer in part (3).

(Part 2) We first show that the maps above are isomorphisms on the resulting rings. Indeed, suppose that $x = (x_0, x_1, \ldots)$ is a sequence in either $\lim_\Phi A^\circ/p$ or $\lim_\Phi A^\circ/\varpi$. Let $(\overline{x}_0, \overline{x}_1, \ldots)$ be an arbitrary lift to a sequence of elements in $\prod_{\mathbb{N}} A^\circ$. Define the inverse map $\ell : \lim_\Phi A^\circ/\varpi \to A^{\flat\circ}$ respectively $\ell : \lim_\Phi A^\circ/p \to A^{\flat\circ}$ via

$$\ell(x)^{(k)} = \lim_{n\to\infty}\overline{x}_{n+k}^{p^n}$$

This limit is well defined since $\varpi \mid p$ implies that in either case, for any $n \geq 0$ and $m' \geq m \geq 0$, we have

$$\overline{x}_{n+m'}^{p^{m'-m}} \equiv \overline{x}_{n+m} \mod \varpi$$

hence $\overline{x}_{n+m'}^{p^{m'}} \equiv \overline{x}_{n+m}^{p^m} \mod \varpi^{m+1}$ and the limit converges in A° for each k by completeness (this also implies that the result lies in $A^{\flat\circ}$). A similar computation also implies that it is independent of the lift $\overline{x}$. Moreover, by definition of the additive structure on $A^\flat$, this map is a ring homomorphism. Finally, each map is an inverse to the quotient map since for any $x \in A^{\flat\circ}$, x is a lift of the projection to $\lim_\Phi A^\circ/p$ respectively $\lim_\Phi A^\circ/\varpi$. Conversely, if $x \in \lim_\Phi A^\circ/p$ or $\lim_\Phi A^\circ/\varpi$, then $\ell(x)^{(k)} \equiv \lim_{n\to\infty} x_{n+k}^{p^n} = x_k \mod p$ respectively ϖ. Finally, since each ring is given the discrete topology, the continuity of the maps in both direction is clear, so that the isomorphism is in fact a topological isomorphism.

(Part 3) Let $\varpi_0 \in A^\circ$ be any pseudo-uniformizer of A such that $\varpi_0^p \mid p$. Consider the map $A^{\flat\circ} = \lim_\Phi A^\circ/\varpi_0^p$ given by projection to the first coordinate. Since the Frobenius map $A^\circ/\varpi_0^p \to A^\circ/\varpi_0^p$ is surjective by definition of a perfectoid ring since ϖ_0^p is a pseudo-uniformizer, we find that the natural map $A^{\flat\circ} \to A^\circ/\varpi_0^p$ is surjective. Thus there exists a $\varpi^\flat$ lifting $\varpi_0 \in A^\circ/\varpi_0^p$. Now consider the map of multiplicative monoids $\sharp : A^\flat \to A$ given by projection to the first coordinate. Let $\varpi = \varpi^{\flat\sharp}$. Note that by definition, ϖ admits a sequence of p-th power roots. Moreover, the image of $\varpi^\flat$ under the composition $A^{\flat\circ} \cong \lim_\Phi A^\circ/\varpi_0^p \to A^\circ/\varpi_0^p$ sends $(\varpi, \varpi^{1/p}, \ldots) \mapsto \varpi \mod \varpi_0^p$. Thus we conclude that $\varpi \equiv \varpi_0 \mod \varpi_0^p$. In other words, ϖ is a pseudo-uniformizer, and the same then holds for $\varpi^\flat$. This then proves the assumption of the previous proposition. Moreover, it implies that $A^\flat$ is Tate finishing the proof of (1).

To show that $A^\flat$ is perfectoid, it suffices to show that $A^{\flat\circ}/\varpi^\flat \cong A^\circ/\varpi$ and that $A^\flat$ is uniform. For the former, consider the induced map of multiplicative monoids $\sharp : A^{\flat\circ} \to A^\circ/\varpi$. Since both rings have characteristic p, the map of multiplicative monoids is in fact a map of rings. Using the isomorphism $A^{\flat\circ}/\varpi^\flat \cong (\lim_\Phi A^\circ/\varpi)/\varpi^\flat$, we may find for any element $\alpha \in A^\circ/\varpi$, a compatible sequence $(\alpha_0, \alpha_1, \ldots) \in A^{\flat\circ}$ such that $\alpha_0 \equiv \alpha \mod \varpi$. This implies surjectivity of the map $A^{\flat\circ} \to A^\circ/\varpi$. Suppose now that $\alpha \in \ker\sharp$. Then $\alpha = (\alpha^{(0)}, \alpha^{(1)}, \ldots)$ such that $\alpha^{(0)} \in (\varpi)$. But then $(\alpha^{(k)})^{p^k} \in (\varpi)$ and hence $\alpha^{(k)} \in (\varpi^{1/p^k})$ (note here that A has no nonzero nilpotents since if x is nilpotent, then $Ax \subset A^\circ$ is unbounded and hence A is not uniform). Thus $\alpha \in (\varpi^\flat)$ and the induced map $A^{\flat\circ}/\varpi^\flat \to A^\circ/\varpi$ is an isomorphism.

To show that $A^\flat$ is uniform, we show that in fact any perfect characteristic p complete Tate ring A is uniform. Indeed, if A_0 is a ring of definition and ϖ a pseudo-uniformizer, write $A_n = A_0^{1/p^n}$ and set $A_\infty = \operatorname{colim} A_n$. Let $f \in A^\circ$. Then $f^{\mathbb{N}}$ is bounded, so $\varpi^a f^{\mathbb{N}} \subset A_0 \subset A_\infty$ for some $a \geq 0$. Since A_∞ is closed under p-th roots, it follows that $\varpi^{a/p^n} f \in A_\infty$ for all $n \geq 0$. Thus $\varpi^{1/p^n} A^\circ \subset A_\infty$ for any $n \in \mathbb{N}$. Similarly, consider the Frobenius map $\mathrm{Frob} : A \to A$. This is a continuous bijection of Banach spaces and thus open. Thus $\varpi^m A_1 \subset A_0$ for some $m \geq 0$ and hence $\varpi^{m/p^n} A_{n+1} \subset A_n$. Thus $\varpi^{\sum_{i=0}^n m/p^i} A_{n+1} \subset A_0$ for all $n \in \mathbb{N}$. Thus we may find an a such that $\varpi^a A_\infty \subset A_0$. Thus there exists an $n \in \mathbb{N}$ such that $t^{1/p^n} A^\circ \subset A_0$, and A° is bounded.

Finally, we must prove that $A^\flat = A^{\flat\circ}[1/\varpi^\flat]$. Indeed, if $\alpha = (\alpha^{(0)}, \alpha^{(1)}, \ldots) \in A^\flat$, then we may write $\alpha^{(0)} = \varpi^k \beta^{(0)}$ with $k \in \mathbb{Z}$ and $\beta^{(0)} \in A^\circ$. Set $\beta^{(n)} = \alpha^{(n)}/\varpi^{k/p^n}$. Then $(\beta^{(n)})^{p^n} = \beta^{(0)} \in A^\circ$, thus $\beta^{(n)} \in A^\circ$ and $\alpha = (\varpi^\flat)^k \beta$ with $\beta = (\beta^{(0)}, \beta^{(1)}, \ldots) \in A^{\flat\circ}$. □