TILTING EQUIVALENCE FOR PERFECTOID ALGEBRAS

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Abstract. We record a proof of the equivalence of categories of perfectoid algebras over a perfectoid field and over its tilt.

1. Almost mathematics

Fix a ring R with an R-flat ideal I such that $I^2 = I$. Let

$$(\mathsf{Mod}_R^a, (-)^a \colon \mathsf{Mod}_R \to \mathsf{Mod}_R^a)$$

be the universal exact functor sending I-torsion modules to 0, defined upto unique isomorphism. The ordinary $\otimes$-product on Mod_R descends to a symmetric monoidal structure on Mod_R^a. For any object M of Mod_R^a, define $M_* := \mathrm{Hom}_{\mathsf{Mod}_R^a}(R^a, M)$. This is endowed with an $\mathrm{End}_{\mathsf{Mod}_R^a} R^a$-module structure by precomposition. After identifying $\mathrm{End}_{\mathsf{Mod}_R^a} R^a$ with R as rings, we get a (lax) symmetric monoidal functor

$$(-)_* \colon \mathsf{Mod}_R^a \to \mathsf{Mod}_R$$

which is right adjoint to $(-)^a$. Define

$$(-)_! := I \otimes_R (-)_* \colon \mathsf{Mod}_R^a \to \mathsf{Mod}_R.$$

It can be checked that this is left adjoint to $(-)^a$.

We now want to restrict to subcategories of commutative algebra objects. It is clear that $(-)^a$ and $(-)_*$ restrict to functors

$$(-)^a \colon \mathsf{CAlg}(\mathsf{Mod}_R) \to \mathsf{CAlg}(\mathsf{Mod}_R^a), \qquad (-)_* \colon \mathsf{CAlg}(\mathsf{Mod}_R^a) \to \mathsf{CAlg}(\mathsf{Mod}_R)$$

forming an adjoint pair. However, $(-)_!$ doesn't restrict because $I \otimes_R M_*$ fails to have a unit morphism. To fix this, we define $(-)_{!!}$ to be the pushout of $R \leftarrow I \simeq (R^a)_! \to (-)_!$. Then the natural map $\mathrm{id}_{\mathsf{Mod}_R} \to (-)_{!!}$ gives funtorial unit, defining a left adjoint

$$(-)_{!!} \colon \mathsf{CAlg}(\mathsf{Mod}_R^a) \to \mathsf{CAlg}(\mathsf{Mod}_R).$$

Of course, this in particular means $(-)_{!!}$ commutes with all colimits.

Proposition 1.1. *$(-)_!$ preserves flatness.*

Proof. M being R^a-flat is equivalent to $\mathrm{Tor}_\bullet^R(M_*, N)^a$ being 0 for all $N \in \mathsf{Mod}_R$. Since tensoring by I kills I-torsion, we get the desired result. □

Proposition 1.2. *$(-)_{!!}$ preserves faithful flatness.*

Proof. Let M be a faithfully flat R^a-algebra. This is equivalent to saying $f \colon R^a \to M$ is injective and M/R^a is R^a-flat. Indeed, for the forward direction, $N \to N \otimes_{R^a} M$ is injective because $N \otimes_{R^a} M \to N \otimes_{R^a} M \otimes_{R^a} M$ has a retraction $n \otimes m \otimes m' \mapsto n \otimes mm'$. Applying the Tor sequence to $0 \to R^a \to M \to M/R^a \to 0$ we

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get that M/R^a is flat. Conversely, if M/R^a is flat then using the same short exact sequence $R^a \to M$ is universally injective, which clearly implies faithful flatness. There is a defining exact sequence

$$0 \longrightarrow I \longrightarrow R \oplus M_! \longrightarrow M_{!!} \longrightarrow 0$$

from which it is clear that $f_{!!}$ is injective as $(-)_!$ is exact. Now, $M_{!!}/R \simeq M_!/I \simeq (M/R^a)_!$, it follows that $M_{!!}/R$ is R-flat by Proposition 1.1, which completes the proof. □

2. Perfectoid algebras

Fix a perfectoid field K with tilt $K^\flat$. Let $t \in K^\flat$ be a pseudouniformizer such that $\pi := t^\sharp$ is a pseudouniformizer for K so that $|p| \leqslant |\pi| < 1$. All almost mathematics is performed with respect to the ideal $K^{\circ\circ}$. Note that we have a distinguished collection of pth-power roots of π.

Definition 2.1.
(a) A **perfectoid K-algebra** is a Banach K-algebra R such that R° is bounded, and the Frobenius $R^\circ/\pi^{1/p} \to R^\circ/\pi$ is surjective.
(b) A **perfectoid $K^{\circ a}$-algebra** A is a π-adically complete flat $K^{\circ a}$-algebra such that $K^{\circ a}/\pi \to A/\pi$ is relatively perfect.
(c) A **perfectoid $K^{\circ a}/\pi$-algebra** A is a flat $K^{\circ a}/\pi$-algebra such that $K^{\circ a}/\pi \to A$ is relatively perfect.

With continuous, adic, and ordinary morphisms, respectively, we get categories $\mathsf{Perf}_K, \mathsf{Perf}_{K^{\circ a}}$, and $\mathsf{Perf}_{K^{\circ a}/\pi}$.

Theorem 2.2 (Tilting equivalence)**.** *Let K be a perfectoid field. The categories of perfectoid K-algebras is equivalent to the category of perfectoid $K^\flat$-algebras. In fact, we have the following chain of equivalences*

$$\mathsf{Perf}_K \simeq \mathsf{Perf}_{K^{\circ a}} \simeq \mathsf{Perf}_{K^{\circ a}/\pi} \simeq \mathsf{Perf}_{K^{\circ\flat a}/t} \simeq \mathsf{Perf}_{K^{\circ\flat a}} \simeq \mathsf{Perf}_{K^\flat}.$$

The first equivalence is given by the pair $(R \mapsto R^{\circ a}, A \mapsto A_*[1/t])$ and the second equivalence is given by reduction modulo π.

As $K^\circ/\pi \simeq K^{\circ\flat}/t$, it suffices to prove the first half of the equivalences. To establish the theorem, we need to construct integral lifts of perfectoid K-algebras R. A natural approach is to 'normalize' K° within R (see Proposition 2.3). By standard almost ring theory yoga, this allows us to transfer information from the generic fiber to the almost integral fiber, leading to the first equivalence of categories. For the second equivalence, we need to construct characteristic 0 lifts (at least when K has characteristic 0) of characteristic p perfectoid algebras. This is achieved using a Witt vector construction.

Proposition 2.3 (Integral models for Banach K-algebras)**.** *Fix a pseudouniformizer $t \in K$. The following categories are equivalent:*

(1) The category $\mathscr{C}$ of uniform Banach K-algebras R with continuous K-algebra maps.
(2) The category $\mathscr{D}_{\mathrm{tic}}$ of π-adically complete and t-torsion-free K°-algebras A which are totally integrally closed in $A[\frac{1}{t}]$.
(3) The category $\mathscr{D}_{\mathrm{prc}}$ of π-adically complete and t-torsion-free K°-algebras A which are p-root closed in $A[\frac{1}{t}]$ and $A \simeq (A^a)_*$.

The equivalence between (1) and (2) is implemented by $R \mapsto R^\circ$ and $A \mapsto A[\frac{1}{t}]$. The equivalence between (2) and (3) is just identity.

Proof. This is routine. See [Bha, Proposition 5.2.5-6]. □

Lemma 2.4 (Nonzero perfectoids are faithfully flat). *Nonzero perfectoid $K^{\circ a}$-algebras and $K^{\circ a}/\pi$-algebras are faithfully flat.*

Proof. Let $R \in \{K^{\circ a}, K^{\circ a}/\pi\}$, and A be a nonzero perfectoid R-algebra. We need to show that $M \otimes_R A = 0$ implies $M = 0$. Let $R \to M$ be an nonzero almost element with kernel I. This gives an injection $R/I \to M$. Therefore, it suffices to show that $A/IA = 0$ implies $R/I = 0$. So if $R \to A$ is not faithfully flat, there must exist a pseudouniformizer ϖ such that $A/\varpi = 0$. Since $\varpi \mid \pi$, this implies $A = 0$, which is a contradiction. □

3. **Proof of the equivalence** $\mathsf{Perf}_K \simeq \mathsf{Perf}_{K^{\circ a}}$

Let $R \in \mathsf{Perf}_K$. We check that $A := R^{\circ a}$ is perfectoid. By Proposition 2.3, we know that R° is π-adically complete, t-torsion-free, and p-root closed in $R[\frac{1}{t}]$. This implies that A is π-adically complete and flat. Since almostification is exact, it suffices to check $\mathrm{Ker}(R^\circ/\pi \xrightarrow{\text{Frob}} R^\circ/\pi) = (\pi^{1/p})$. This is easy– if $x \in R^\circ$ is such that $x^p \in (\pi)$ then $\pi^{-1/p}x \in R$ has pth power in R° which by definition of powerboundedness implies $\alpha \in (\pi^{1/p})$.

For the other direction, by Proposition 2.3, we must check that given $A \in \mathsf{Perf}_{K^{\circ a}}$, A_* is π-adically complete, t-torsion-free, p-root-closed in $A_*[\frac{1}{\pi}]$, and has a surjective Frobenius modulo π. Torsion-freeness and t-adic completeness are immediate by the following lemma.

Lemma 3.1. *Let M be a $K^{\circ a}$-module.*

(1) M is flat if and only if M_* is flat if and only if M_* has no t-torsion.
(2) If N is a flat K°-module such that $M = N^a$, then $M_* = \{x \in N[\frac{1}{t}] : K^{\circ\circ}x \in N\}$.
(3) If M is flat then for all $x \in K^\circ$, we have $(xM)_* = xM_*$. Further, $M_*/xM_* \subset (M/xM)_*$.
(4) If M is flat then the image of $(M/x\varepsilon M)_*$ in $(M/xM)_*$ is equal to M_*/xM_* for all $\varepsilon \in K^{\circ\circ}$. In particular, M is π-adically complete if and only if M_* is π-adically complete.

Proof. (1)-(3) are more or less obvious– these are proved using the fact that flatness is equivalent to t-torsion-freeness, and left/right-exactness of various functors coming from adjunction relations. See [Sch12, Lemma 5.3] or [Bha, Lemma 4.4.1] for more details. For (4), If $m \in (M/xM)_*$ lifts to $\tilde{m} \in (M/x\epsilon M)_*$, then $\tilde{m}(\epsilon) = n \in M_*/x\epsilon M_*$, which lifts to $\tilde{n} \in M_*$. We check $\tilde{n}$ is divisible by ϵ, as $\delta\tilde{n} \in \epsilon M_*$ for any $\delta \in m$. Hence, $\tilde{n} = \epsilon m_1$ for $m_1 \in M_*$, which is the desired lift of m. Multiplication by ϵ induces an injection $(M/xM)_* \to (M/x\epsilon M)_*$, as $(-)_*$ is left-exact, and $\tilde{n}$ maps to $\epsilon m = n$ in $(M/x\epsilon M)_*$. Since $(-)^a$ commutes with limits and colimits, so M_* complete $\implies$ M complete is trivial. Conversely, assume M is flat and π-adically complete. Then $M_* \simeq (\lim M/t^nM)_* \simeq \lim (M/t^nM)_* \simeq \lim M_*/t^nM_*$. □

For p-root closedness: there is a commutative square

$$\begin{array}{ccc} (A/\pi^{1/p})_* & \xrightarrow{\text{Frob}_*} & (A/\pi)_* \\ \uparrow & & \uparrow \\ A_*/\pi^{1/p} & \xrightarrow{\text{Frob}} & A_*/\pi \end{array}$$

The upper horizontal arrow is an isomorphism. Therefore, the lower arrow is an injection– if $y \in A_*$ satisfies $y^p \in \pi A_*$, then $y \in \pi^{1/p}A_*$. Fix some $x \in A_*[\frac{1}{\pi}]$ with $x^p \in A_*$. Then $y := \pi^{k/p}x$ lies in A_* for some large enough positive integer k. Taking pth powers, $y^p = \pi^k x^p \in \pi^k A_* \in \pi A_*$, which implies $y \in \pi^{1/p}A_*$. Therefore, $\pi^{(k-1)/p}x \in A_*$. Iterating this, we get $x \in A_*$.

For surjectivity of Frobenius modulo π: the Frobenius is almost surjective modulo π by assumption, therefore it is enough to show that the Frobenius on $A_*/K^{\circ\circ}A_*$ is surjective. Take some $x \in A_*$ and arbitrary $c < 1$. Almost surjectivity implies $\pi^c x \equiv y^p \pmod{\pi A_*}$ for some $y \in A_*$. Then by p-root-closedness, $z := \pi^{-c/p} y$ lies in A_*. Thus, $y \in \pi^{c/p} A_*$ This implies $x \equiv z^p \pmod{\pi^{1-c} A_*}$. Since $\pi^{1-c}A_* \subset K^{\circ\circ}$, the claim follows.

To check that they are inverses: observe $(R^{\circ a})_*[1/t] \simeq R^\circ[1/t] = R$ and $(A_*[1/t])^{\circ a} \simeq (A_*)^a = A$. □

4. Proof of the equivalence $\mathsf{Perf}_{K^{\circ a}} \simeq \mathsf{Perf}_{K^{\circ a}/\pi}$

There is an obvious "mod π reduction" functor $\mathsf{Perf}_{K^{\circ a}} \to \mathsf{Perf}_{K^{\circ a}/\pi}$. By topological invariance of formally étale algebras (c.f. [Bha, Theorem 6.1.3]), there is an equivalence of categories:

$$\mathscr{C} := \left\{ \begin{matrix} \pi\text{-adically complete, flat } K^\circ\text{-algebras with} \\ \text{surjective Frobenius mod } \pi \end{matrix} \right\} \xrightarrow{(-)\otimes_{K^\circ} K^\circ/\pi} \mathscr{C}_0 := \left\{ \begin{matrix} \text{Flat } K^\circ/\pi\text{-algebras with} \\ \text{surjective Frobenius} \end{matrix} \right\}$$

Take some $A \in \mathsf{Perf}_{K^{\circ a}/\pi}$. We want to lift A to an almost integral perfectoid algebra. We may assume A is nonzero so that it is faithfully flat. Then $A_{!!}$ is a faithfully flat K°/π-algebra. It has surjective Frobenius because $(-)_{!!}$ preserves colimits. By the above equivalence of categories, we have an integral lift $\widetilde{A_{!!}} \in \mathscr{C}$. We claim that the functor $A \mapsto \overline{A} := (\widetilde{A_{!!}})^a$ is the required inverse. Since almostification is exact and preverses limits, colimits, it follows that $\overline{A}$ is $K^{\circ a}$-flat, π-adically complete, has surjective Frobenius mod π. Thus, $\overline{A}$ is perfectoid. Observe that $\overline{A}/\pi \simeq (\widetilde{A_{!!}}/\pi)^a \simeq (A_{!!})^a \simeq A$. For the other direction, we must check $A \simeq \overline{A/\pi}$ for $A \in \mathsf{Perf}_{K^{\circ a}}$. Assume A is nonzero. Unravelling the definitions, $\overline{A/\pi} = (\widetilde{(A/\pi)_{!!}})^a \simeq (\widetilde{A_{!!}/\pi})^a$. It therefore suffices to show that $A_{!!}$ is the unique integral perfectoid lift of $A_{!!}/\pi$, which in turn is equivalent to showing $A_{!!}$ is π-adically complete. There is a canonical map $A_* \to A_{!!}$ which is an almost isomorphism. Due to torsion-freeness, this must hence be an injection with almost zero cokernel. The proof is now complete by the following lemma:

Lemma 4.1. *Let A be a ring and $t \in A$ a nonzerodivisor. Let $M \to N$ be an injective map of t-torsion-free A-modules with cokernel killed by t. Then M is t-adically complete if and only if N is so.*

Proof. Form an exact sequence

$$0 \to M \to N \to Q \to 0.$$

Since $\mathrm{Tor}_1^A(A/t^n, Q) = Q[t^n]$ and $\mathrm{Tor}_1^A(A/t^n, N) = 0$, we get exact sequences

$$0 \to Q[t^n] \to M/t^n \to N/t^n \to Q \to 0$$

for all n. Taking inverse limits, the claim follows from the following monomorphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widehat{M} & \longrightarrow & \widehat{N} & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

□

References

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