TATE UNIFORMIZATION OF DRINFELD MODULES

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Abstract. We give an account of Tate uniformization of Drinfeld modules.

1. Introduction

Let us fix the following notations:

- K a local field with integer ring $(\mathscr{O}, \mathfrak{m}, k)$.
- $A = \mathbb{F}_q[T]$, $F = \mathbb{F}_q(T), F_\infty = \mathbb{F}_q[T^{-1}]\llbracket T \rrbracket$
- $\mathbb{C}_\infty$ the completed algebraic closure of $\mathbb{F}_\infty$.
- for any characteristic p ring R, the *skew-polynomial ring* $R\{\tau\}$ is defined as the endomorphism ring $\operatorname{End} \mathbb{G}_{a,R}$ where τ denotes the Frobenius endormorphism.
- similarly, $R\{\{\tau\}\}$ is the *skew-power-series ring*.

Definition 1.1. Let $\mathbf{E}$ be a Drinfeld module over $\mathscr{O}$. A lattice of rank d in $\mathbf{E}$ is an injective A-module homomorphism $\nu \colon \Lambda \to \mathbf{E}(K^{\text{sep}})$, where Λ is a free A-module of rank d, whose image is discrete and invariant under the action of $\operatorname{Gal}(K^{\text{sep}}/K)$.

Definition 1.2. A **Tate datum** of rank (d_1, d_2) over $\mathscr{O}$ is a pair $(\mathbf{E}, \Lambda)$, where $\mathbf{E}$ is a Drinfeld module of rank d_1 over $\mathscr{O}$ and $\Lambda = (\Lambda, \nu)$ is a lattice in $\mathbf{E}$ of rank d_2. For Tate data $(\mathbf{E}, \Lambda, \nu)$ and $(\mathbf{E}', \Lambda', \nu')$ of same rank, a morphism is a commutative square

$$\begin{tikzcd}
\Lambda \arrow[r, "\nu"] \arrow[d, "\Phi"] & \mathbf{E}(K^{\text{sep}}) \arrow[d, "\varphi(K^{\text{sep}})"] \\
\Lambda' \arrow[r, "\nu'"] & \mathbf{E}'(K^{\text{sep}})
\end{tikzcd}$$

where Φ is a A-module map and φ is a morhpism of Drinfeld $\mathscr{O}$-modules.

The main theorem of Tate uniformization is the following

Theorem 1.3. *Let d be a positive number. The category of Tate data $(\mathbf{E}, \Lambda)$ where $\mathbf{E}$ is a rank r Drinfeld $\mathscr{O}$-module with good reduction and Λ is a rank d lattice is equivalent to the full subcategory of Drinfeld modules of rank $r + d$ over $\mathscr{O}$ with stable reduction of rank r.*

2. Quotienting by a lattice

For $\omega \in \mathbb{Q}_p$ with $|\omega| < 1$, the Tate curve is a rigid analytic elliptic curve defined as $\mathbb{Q}_p/\omega^{\mathbb{Z}}$. We will try to imitate this construction in the case of Drinfeld modules. Let $\phi \colon A \to K\{\tau\}$ be a Drinfeld module and Λ

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be a lattice. Define

$$\exp_\Lambda(x) = x \prod_{\lambda\in\Lambda\setminus\{0\}} \left(1 - \frac{x}{\lambda}\right).$$

This is an $\mathbb{F}_q$-linear entire function with coefficients in K because Λ is discrete and Galois stable. Let $\Lambda' = \phi_a^{-1}\Lambda \subset \mathbb{C}_K$. Note that Λ' is not necessary free because $\phi[a] \subset \Lambda'$. The following exact sequence is easily checked

$$0 \to \phi[a] \to \Lambda'/\Lambda \to \Lambda/\phi_a\Lambda \to 0. \tag{1}$$

In particular, Λ has finite index in Λ', and consequently Λ' is discrete. Observe that the zero set of $\exp_\Lambda \circ \phi_a$ is Λ'. Thus, $a\exp_{\Lambda'}(x) = \exp_\Lambda(\phi_a(x))$ by using the characteristic property of exponential functions and comparing constant coefficients. On the other hand, there exists an $\mathbb{F}_q$-linear polynomial P_a such that $\exp_{\Lambda'} = P_a \circ \exp_\Lambda$ (see [Pap, Lemma 5.1.4]). Denoting $\psi_a(x) = aP_a(x)$, we have

$$\exp_\Lambda \circ \phi_a = \psi_a \circ \exp_\Lambda.$$

Since $\exp_\Lambda$ and ϕ_a are defined over K, the same is true for ψ_a. The following is now clear:

Proposition 2.1. *With the above notations, the map $\psi\colon A \to K\{\tau\}$, $a \mapsto \psi_a$ is a Drinfeld module over K of rank $r+d$, where $r = \operatorname{rank}\phi$ and $d = \operatorname{rank}\Lambda$.*

In summary, we have the exact sequence

$$0 \to \Lambda \to \phi(\mathbb{C}_K) \xrightarrow{\exp_\Lambda} \psi(\mathbb{C}_K) \to 0,$$

which is called the **Tate uniformization of** ψ if ϕ acquires good reduction.

3. **Standard endomorphisms**

Lemma 3.1. *Let B be an $\mathbb{F}_q$-algebra. Suppose $u = \sum_{i=0}^n u_i\tau^i \in B\{\tau\}$ is such that u_0 is invertible and $u_1,\dots,u_n$ are nilpotent. Then u is invertible.*

Proof. Straightforward. Omitted. □

Lemma 3.2. *Let B be an $\mathbb{F}_q$-algebra and let $d > 0$ be an integer. Suppose $f = \sum_{i=0}^n f_i\tau^i \in B\{\tau\}$ is such that $f_d \in B^\times$ and $f_{d+1},\dots,f_n$ are nilpotent. Then there exists a unique $u = \sum_{j\geqslant 0} u_j\tau^j \in B\{\tau\}$ such that $u_0 = 1$, u_j are nilpotent for $j \geqslant 1$, and $g = u^{-1}fu = \sum_{i=1}^d g_i\tau^i$ has degree d and $g_d \in B^\times$. Such polynomials are called **standard**.*

Proof. Let $N = (f_{d+1},\dots,f_n)$. There exists some minimal positive integer k such that $N^k = 0$. We induct on k. If $k = 1$ then there is nothing to do. Suppose the result is true for all $k \leqslant s$. By induction hypothesis applied to B/N^{s-1} and the image of f therein, there exists u' and g' in $B\{\tau\}$ which solve the problem modulo N^{s-1}. It is easy to see that g'_d is a unit and the ideal $I = (g'_{d+1},\dots,g'_m)$ lies in N^{m-1}. Hence, $I^2 = 0$. Let

$$f' = \left(1 - \frac{f_n}{f_d^{q^n-d}}\tau^{n-d}\right) \circ f \circ \left(1 - \frac{f_n}{f_d^{q^n-d}}\tau^{n-d}\right)^{-1}.$$

By computing, it can be seen that f' has leading term $b\tau^m$ where $m < n$. Using induction, this shows the existence of u. If v is another polynomial satisfying the specified conditions, then consider $h = vu^{-1}$. We have $h(ufu^{-1}) = (vfv^{-1})h$, from where it is just a matter of comparing coefficients to derive $h = 1$. □

Lemma 3.3. *Let $f \in \mathscr{O}\{\tau\}$ with $d = \deg\overline{f} > 0$, where $\overline{f}$ denotes the reduction of f modulo $\mathfrak{m}\{\tau\}$. There exists a unique $u \in R\{\{\tau\}\}$ such that $u = 1 + \sum_{i\geqslant 1}\alpha_i\tau^i$, $|\alpha_i| < 1$, $\alpha_i \to 0$, $g = u^{-1}fu$ lies in $R\{\tau\}$, and $\deg g = \deg\overline{g} = d$, and u is an entire function.*

Proof. By applying Lemma 3.2 to to $\mathcal{O}/\mathfrak{m}^m$, $m \geqslant 1$, we can form $u \in R\{\{\tau\}\}$ such that $\partial u = 1$ and $g = u^{-1} f u$. It's clear that $\deg g = \deg \overline{g} = d$ and that $|\alpha_i| < 1$ because α_i is nilpotent modulo $\mathfrak{m}^{i+1}$, and in particular belongs to $\mathfrak{m}$. Also, $\alpha_i \to 0$ because $u_i \equiv u_{i-1} \pmod{\mathfrak{m}^{i-1}}$. It remains to show that $u(x)$ is entire. For $m > n$, comparing the coefficient of τ^m on both sides of $ug = fu$ gives

$$\alpha_m g_0^{q^m} + \alpha_{m-1} g_1^{q^{m-1}} + \cdots + \alpha_{m-d} g_d^{q^{m-d}} = \alpha_m f_0 + \alpha_{m-1}^q f_1 + \cdots + \alpha_{m-n}^{q^n} f_n,$$

which can be rewritten as

$$\alpha_{m-d}(g_d^{q^{m-d}} - f_d \alpha_{m-d}^{q^d-1}) = -\sum_{i=0}^{d-1} \alpha_{m-i} g_i^{q^{m-i}} + \sum_{\substack{0 \leqslant j \leqslant n \\ j \neq d}} f_j \alpha_{m-j}^{q^j}.$$

Here g_d is a unit. Taking absolute values and applying triangle inequality,

$$|\alpha_{m-d}| \leqslant \max\{|\alpha_{m-d+1}|, \ldots, |\alpha_m|, |\alpha_{m-n}|^{q^n}, |\alpha_{m-n+1}|^{q^{n-1}}, \ldots, \widehat{|\alpha_{m-d}|^{q^d}}, \ldots, |\alpha_m|\},$$

where $\widehat{|\alpha_{m-d}|^{q^d}}$ means that the term has been removed. Since $|\alpha_{m-j}| < 1$ for $0 \leqslant j \leqslant d-1$, we have $|\alpha_{m-j}|^{q^j} < |\alpha_{m-j}|$. Therefore, for $i > s := n - d$

$$|\alpha_i| \leqslant \max\{|\alpha_{i-s}|^{q^{d+s}}, \ldots, |\alpha_{i-1}|^{q^{d+1}}, |\alpha_{i+1}|, \ldots, |\alpha_{i+d}|\}.$$

Denote the set on the right-hand side of this inequality by S_i. Now execute the following iterative process. Initially, put $S := S_i$. If $|\alpha_i|^{q^\ell} \in S$ for some $\ell \geqslant 1$, then delete that element from S. Next, replace each $\left|\alpha_j\right|^{q^\ell} \in S$ with $j > i$ by $S_j^{q^\ell}$, where $S_j^{q^\ell}$ denotes the set of elements of S_j raised to power q^ℓ; call the resulting set S. Repeat the same process for this new S. It is easy to see that with each iteration, either the elements $\left|\alpha_j\right|$ appear in S to higher powers of q than before or $\left|\alpha_j\right|$ has larger index than the elements in the previous S. At each step of the process we have $|\alpha_i| \leqslant \max S$. On the other hand, since $0 \leqslant \left|\alpha_j\right| < 1$ for all $j > 0$ and $\left|\alpha_j\right| \to 0$ as $j \to \infty$, the maximum of the elements in S with indices greater than i will tend to 0 . Therefore,

$$|\alpha_i| \leqslant \max\left(|\alpha_{i-s}|^{q^{d+s}}, \ldots, |\alpha_{i-1}|^{q^{d+1}}\right).$$

If we denote $\beta_j = \left|\alpha_j\right|^{1/q^j}$, $j \geqslant 1$, then the above implies $\beta_i \leqslant \max(\beta_{i-s}, \ldots, \beta_{i-1})^{q^d}$. From here, one can show that $\beta_{i+2} \leqslant \max\{\beta_{i-s}^{q^d}, \beta_{i+1-s}^{q^d}, \ldots, \beta_{i-1}\}^{q^d}$ and so on, eventually obtaining $\beta_{i+js} \leqslant \max\{\beta_{i-s}, \ldots, \beta_{i-1}\}^{q^{(j+1)d}}$ for all $j \geqslant 0$. Since $\max\{\beta_{i-s}, \ldots, \beta_{i-1}\} < 1$, it follows that $\beta_j \to 0$. $\square$

4. **Proof of Theorem 1.3**

Let $\phi\colon A \to \mathcal{O}_K\{\tau\}$ be a Drinfeld module with good reduction with $\phi_T = T + g_1\tau + \cdots + g_r\tau^r$, $g_r \in \mathcal{O}_K^\times$. Then the construction of Proposition 2.1 gives the desired Drinfeld module. Conversely, suppose we are given a Drinfeld module ψ of rank $r + d$ over $\mathcal{O}_K$ so that its reduction $\overline{\psi}$ has rank r. By Lemma 3.3, we get a unique $e = 1 + \sum_{i=1}^\infty \alpha_i \tau^i \in 1 + \mathfrak{m}\{\{\tau\}\}\tau$, such that $\phi_T = e^{-1}\psi_T e \in \mathcal{O}_K\{\tau\}$ has degree r, $\overline{\phi}_T = \overline{\psi}_T$ and e is entire. The roots of u form a lattice Λ in $\psi(K^{\mathrm{sep}})$. Indeed, it is easy to see that Λ is discrete. Further, any zero λ must satisfy $|\lambda| > 1$, for if $|\lambda| \leqslant 1$, then $e(\lambda) \in 1 + \mathfrak{m}$, which cannot be zero. We now have to show that Λ is a lattice of rank d. By comparing ranks of the terms of the exact sequence

$$0 \to \phi[a] \to \psi[a] \to \Lambda/\phi_a\Lambda \to 0,$$

we get $\Lambda/\phi_a\Lambda \cong (A/aA)^d$. Choose a ball B centered at 0 of suitable radius such that the map $B \cap \Lambda \to \Lambda/\phi_a\Lambda$ is surjective. Since $|ax| > |x|$ for each nonzero x, it follows that Λ is generated by the LHS. This shows that Λ is A-free of rank d.

Next, let $\psi\colon \mathbf{E} \to \mathbf{E}'$ be a morphism of Drinfeld modules with stable reduction of same rank. If $\psi \neq 0$ then $\mathbf{E}$ and $\mathbf{E}'$ have Tate data $(\mathbf{F}, \Lambda)$ and $(\mathbf{F}', \Lambda')$ of the same rank, ψ is defined over $\mathscr{O}_K$ and ψ is not $0 \pmod{\mathfrak{m}}$. Then $\varphi := (u')^{-1}(\psi(u))$ defines an isogeny $\mathbf{F} \to \mathbf{F}'$. It is clear that φ induces a morphism $\Lambda \to \Lambda'$. Conversely, let $(\varphi, \Phi)\colon (\mathbf{F}, \Lambda) \to (\mathbf{F}', \Lambda')$ be a morphism of Tate data. Put $\psi := u_\Lambda(\varphi((u'_\Lambda)^{-1}))$. It is then a matter of checking that this is a polynomial. $\square$

References

[Pap] M. Papikan, *Drinfeld Modules*, Graduate Texts in Mathematics **296**, Springer Cham, 2023