SPECIAL CYCLES ON UNITARY SHIMURA VARIETIES

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Abstract. This is a brief introduction to Kudla's generating series of special cycles on unitary Shimura varieties and statements of modularity.

1. Unitary Shimura varieties

1.1. Setup. Let F be a totally real number field of degree $d > 1$ and F a CM-extension of F. Let $(V, \langle\cdot,\cdot\rangle)$ be a Hermitian n-dimensional F-vector space whose inner product has signature $(n-1,1)$ at a distinguished archimedean place $\iota\colon F \hookrightarrow \mathbb{R}$ and signature $(n,0)$ at all other archimedean places. Such an inner product space is called *standard indefinite*. For a place v, we denote $V_v = V \otimes_F F_v$. Let Herm_m be the closed subscheme of $\mathrm{Res}_{E/F}\mathrm{Mat}_{m\times m}$ classifying $m \times m$ Hermitian matrices. A complex line spanned by a vector $v \in V_\iota$ is called *negative-definite* if $\langle v, v\rangle < 0$.

1.2. Shimura datum. Let $H = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{U}(V)$, where $\mathrm{U}(V)$ denotes the unitary group. Define

$$h : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \longrightarrow H_{\mathbb{R}} \cong \mathrm{U}(n-1,1)_{\mathbb{R}} \times \mathrm{U}(n,0)_{\mathbb{R}}^{d-1}$$
$$h(z) = \left(\begin{pmatrix} \mathbb{1}_{n-1} & \\ & \bar{z}/z \end{pmatrix}, \mathbb{1}_n, \dots, \mathbb{1}_n\right).$$

Let D be the $H(\mathbb{R})$-conjugacy class of h, defined as $H(\mathbb{R})/\mathrm{Stab}_{H(\mathbb{R})}(h)$ where $\mathrm{Stab}_{H(\mathbb{R})}(h)$ is the stabilizer of h. Noncanonically, it is easily seen that $D \cong \mathrm{U}(n-1,1)/(\mathrm{U}(n-1)\times\mathrm{U}(1))$. Note that $\mathrm{U}(n-1,1)$ acts on the set of all negative-definite complex lines in V_ι transitively with stabilizer of a line being $\mathrm{U}(n-1)\times\mathrm{U}(1)$. Thus, D can be identified with an open subset of $(\mathbb{P}V_\iota)^{\mathrm{an}}$, classifying negative-definite $\mathbb{C}$-lines, where the complex structure is given by a choice of a complex place of E above ι. Then D is what's called a *Hermitian symmetric domain* for $H(\mathbb{R})$. The pair (G, D) forms a *Shimura datum*.

1.3. Shimura manifolds (and varieties). For any open compact subgroup K of $H(\mathbb{A}_{\mathbb{Q}}^{\infty})$, we can form the double coset space

$$X_K := H(\mathbb{Q})\backslash[D \times H(\mathbb{A}_{\mathbb{Q}}^{\infty})]/K$$

where $H(\mathbb{Q})$ acts diagonally on $D \times H(\mathbb{A}_{\mathbb{Q}}^{\infty})$ on the left, and K acts on $H(\mathbb{A}_{\mathbb{Q}}^{\infty})$ on the right. It comes from a general fact that $H(\mathbb{Q})\backslash H(\mathbb{A}_{\mathbb{Q}}^{\infty})/K$ is finite. So, after choosing coset representives

$$H(\mathbb{A}_{\mathbb{Q}}^{\infty}) = \bigsqcup_i H(\mathbb{Q})h_i K,$$

we can write

$$X_K = \bigsqcup_i \Gamma_i\backslash D,$$

where $\Gamma_i = H(\mathbb{Q}) \cap h_i K h_i^{-1}$. If each Γ_i acts freely on D then X_K is a manifold. This is the case when K is a *neat* open compact subgroup of $H(\mathbb{A}_{\mathbb{Q}}^{\infty})$.

Fact. *The inverse system of complex manifolds $\{X_K\}_K$, with K varying among neat open compact subgroups of $H(\mathbb{A}_{\mathbb{Q}}^{\infty})$, admit "canonical" models $\mathrm{Sh}(H,D) := \{\mathrm{Sh}_K(H,D)\}_K$ as quasi-projective varieties over $E \subset \mathbb{C}$ (a complex place lying above ι). Moreoever, the covering maps $X_K \to X_{K'}$, when $K \subset K'$, are given by complex*

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analytifications of finite étale covers $\mathrm{Sh}_K(H,D) \to \mathrm{Sh}_{K'}(H,D)$ *defined over* E*. Further, all the* $\mathrm{Sh}_K(H,D)$ *are projective if and only if* V *satisfies* $\langle x,x\rangle = 0 \iff x = 0$.

Remark 1.3.1. If $F \neq \mathbb{Q}$ then the last condition is satisfied due to the signature conditions.

1.4. **Hecke action.** We remark that $H(\mathbb{A}_{\mathbb{Q}}^{\infty})$ has a natural right action on the inverse system $\{\mathrm{Sh}_K(H,D)\}_K$ induced by

$$D \times H(\mathbb{A}_{\mathbb{Q}}^{\infty}) \to D \times H(\mathbb{A}_{\mathbb{Q}}^{\infty}),\ (x,s) \mapsto (x,sh)$$

for each $h \in H(\mathbb{A}_{\mathbb{Q}}^{\infty})$ which maps $\mathrm{Sh}_{hKh^{-1}(H,D)}$ to $\mathrm{Sh}_K(H,D)$. These are called *Hecke actions* and they are E-isomorphisms.

2. Special cycles

2.1. **Tautological line bundle.** There is a tautological line bundle $V_\iota \setminus \{0\} \to (\mathbb{P}V_\iota)^{\mathrm{an}}$, whose restriction over D we denote by $\mathscr{L}$. The $H(\mathbb{R})$-action on D lifts naturally to $\mathscr{L}$ and therefore $\mathscr{L}$ descends to a holomorphic line bundle $\mathscr{L}_K$ on X_K given by $H(\mathbb{Q})\backslash\mathscr{L} \times H(\mathbb{A}_{\mathbb{Q}}^{\infty})/K \to X_K$. This line bundle is algebraic and defined over E.

2.2. **Weighted special cycles.** For brevity, we write $X_K = \mathrm{Sh}_K(H,D)$. Let $x \in V$ with $\langle x,x\rangle > 0$ we can form its orthogonal complement $V_x \subset V$. Then V_x is standard indefinite of dimension $n-1$. Let $H_x = \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{U}(V_x) = \mathrm{Stab}_H\, x$ and D_x be the associated Hermitian symmetric domain, which can be identified with $D_x = \{z \in D \colon z \subseteq V_x\}$. Then (H_x, D_x) forms a Shimura datum and the embedding of Shimura datum $(H_x,D_x) \to (H,D)$ induces a Hecke-equivariant morphism

$$\mathrm{Sh}_{K\cap H_x(\mathbb{A}_{\mathbb{Q}}^{\infty})}(H_x,D_x) \to X_K.$$

It can be checked, say on complex points, that this is a closed embedding of codimension 1.

Definition 2.2.1. Denote the image of the above morphism by $Z(x)_K$. We call

$$Z(x)_K \in \mathrm{CH}^1(X_K)$$

a **special divisor**. More generally, for a r-tuple of vector $\mathbf{x} = (x_1,\dots,x_r) \in V^r$ with positive-definite E-span, denoted $\underline{\mathbf{x}}$, we can form $H_{\mathbf{x}} = \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{U}(\underline{\mathbf{x}}^{\perp})$ and $D_{\mathbf{x}} = \{z \in D \colon z \subseteq \underline{\mathbf{x}}^{\perp}\}$ giving rise to a **special cycle**

$$Z(\mathbf{x})_K := \mathrm{Im}(\mathrm{Sh}_{K\cap H_{\mathbf{x}}(\mathbb{A}_{\mathbb{Q}}^{\infty})}(H_{\mathbf{x}},D_{\mathbf{x}}) \hookrightarrow X_K) \in \mathrm{CH}^{\dim \underline{\mathbf{x}}}(X_K).$$

For an element $h \in H(\mathbb{A}_{\mathbb{Q}}^{\infty})$, the **Hecke-translated special cycle** is given by

$$Z(\mathbf{x},h)_K := \mathrm{Im}(\mathrm{Sh}_{H_{\mathbf{x}}(\mathbb{A}_{\mathbb{Q}}^{\infty})\cap hKh^{-1}}(H_{\mathbf{x}},D_{\mathbf{x}}) \hookrightarrow X_{hKh^{-1}} \xrightarrow{\text{Hecke}} X_K) \in \mathrm{CH}^{\dim \underline{\mathbf{x}}}(X_K).$$

For a r-tuple $\mathbf{x} = (x_i) \in V^r$, define the moment matrix $\langle \mathbf{x},\mathbf{x}\rangle = \frac{1}{2}[\langle x_i, x_j\rangle]$. Fix a Hermitian matrix $T \in \mathrm{Herm}_r(F)$ and a K-invariant Schwartz function $\varphi \in \mathscr{S}(V^r \otimes_F \mathbb{A}_F^{\infty})$ for the left regular representation of $H(\mathbb{A}_{\mathbb{Q}}^{\infty})$ on $(V \otimes_F \mathbb{A}_F^{\infty})^r$. Suppose that $\langle \mathbf{x},\mathbf{x}\rangle = T$. Set $\Omega_T = \{\mathbf{v} \in V^r \otimes_F \mathbb{A}_E^{\infty} \colon \langle \mathbf{v},\mathbf{v}\rangle = T\}$. Then

$$\Omega_T \cap \mathrm{Supp}\,\varphi = \bigsqcup_j K h_j^{-1}\mathbf{x}$$

with $h_j \in H(\mathbb{A}_{\mathbb{Q}}^{\infty})$. The disjoint union is finite because the K-orbits give an open cover of the compact set on the left-hand side.

Definition 2.2.2. For a fixed $T \in \mathrm{Herm}_r(F)$ and a K-invariant Schwartz function $\varphi \in \mathscr{S}((V \otimes_F \mathbb{A}_F^\infty)^r)$ we define the **weighted special cycle** as

$$Z'(T,\varphi)_K := \sum_{\substack{j \\ \langle \mathbf{x},\mathbf{x}\rangle = T}} \varphi(h_j^{-1}\mathbf{x}) Z(\mathbf{x}, h_j)_K \in \mathrm{CH}^{\mathrm{rank}\, T}(X_K) \otimes \mathbb{C}.$$

The weighted special cycle has codimension rank T. To correct this, we take an intersection product with a suitable power of the first Chern class of the dual of the tautological line bundle $\mathscr{L}_K^\vee$.

Definition 2.2.3. For a weighted special cycle $Z'(T,\varphi)_K$, the corresponding **normalized weighted special cycle** is defined as

$$Z(T,\varphi)_K := Z'(T,\varphi)_K \cdot c_1(\mathscr{L}_K^\vee)^{r-\mathrm{rank}\, T} \in \mathrm{CH}^r(X_k) \otimes \mathbb{C},$$

where $c_1(\mathscr{L}_K^\vee)$ is the first Chern class of $\mathscr{L}_K^\vee$.

It turns out that this behaves well under pullback. Namely, if $K' \subset K$ and $p_{K',K}\colon X_{K'} \to X_K$ is the natural étale cover, then

$$p_{K',K}^* Z(T,\varphi)_K = Z(T,\varphi)_{K'}.$$

We thus get a well-defined element of $\mathrm{CH}^r(X)_\mathbb{C} := \varinjlim_K \mathrm{CH}^r(X_K) \otimes \mathbb{C}$, so we may reasonably drop the subscript K from the notation.

3. **Kudla's generating series**

Let $G = \mathrm{U}(r,r)$ over F, preserving the skew-Hermitian form

$$J_r = \begin{bmatrix} 0 & \mathbb{1}_r \\ -\mathbb{1}_r & 0 \end{bmatrix},$$

with associated Hermitian symmetric domain

$$\mathscr{H}_r := \{\tau = (\tau_v)_v = (x_v + i y_v)_v \colon (x_v)_v, (y_v)_v \in \mathrm{Herm}_r(\mathbb{R}),\ y_v \text{ is positive-definite for all } v \in \infty_F\},$$

where ∞_F denotes the set of all real places of F. This is called the **Hermitian upper half-space of genus** r.

Definition 3.1. For a K-invariant Schwartz function $\varphi \in \mathscr{S}(V^r \otimes_F \mathbb{A}_F^\infty)$, define **Kudla's generating series of codimension-r special cycles** or **arithmetic theta series** as the formal power series

$$Z(\tau,\varphi)_K := \sum_{T \in \mathrm{Herm}_r(F)} Z(T,\varphi)_K q^T$$

for $\tau \in \mathscr{H}_r$ and

$$q^T := \exp\left(2\pi i \sum_{v \in \infty_F} \mathrm{Tr}(T\tau_v)\right).$$

Remark 3.2. This generating function behaves well under pullback and hence defines a formal power series with coefficients in $\mathrm{CH}^r(X)_\mathbb{C}$.

4. Modularity

For any linear functional $\ell\colon \mathrm{CH}^r(X_K)_{\mathbb{C}} \to \mathbb{C}$, one can form the formal power series $\ell(Z(\tau,\varphi)_K)$ with complex coefficients and ask whether it converges absolutely and is a holomorphic Hermitian modular form on $\mathscr{H}_r$.

Conjecture 4.1 (Arithmetic modularity). *The power series $\ell(Z(\tau,\varphi)_K)$ converges absolutely for any linear functional $\ell\colon \mathrm{CH}^r(X_K)_{\mathbb{C}} \to \mathbb{C}$ and is a Hermitian modular form on $\mathscr{H}_r$.*

We add that [Liu11a, Theorem 3.5] has shown that if it converges absolutely, then it also implies that it is modular.

There is a cycle class map $\mathrm{CH}^r(X_K) \to \mathrm{H}^{2r}(X_K(\mathbb{C}),\mathbb{Z})$, $Z \mapsto [Z]$, given by viewing cycles as defining linear functionals on the space of compactly supported closed forms. Using this, one can define a **geometric theta series** $[Z(\tau,\varphi)_K]$ as a formal power series with coefficients in $\mathrm{H}^{2r}(X_K(\mathbb{C}),\mathbb{C})$. The classical theorem of Kudla-Milson shows that this is indeed absolutely convergent and modular.

Theorem 4.2 (Kudla-Milson). *The power series $\ell([Z(\tau,\varphi)_K])$ converges absolutely for any linear functional $\ell\colon \mathrm{H}^{2r}(X_K(\mathbb{C}),\mathbb{C}) \to \mathbb{C}$ and is a Hermitian modular form on $\mathscr{H}_r$.*

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