Mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$

Ayan Nath

ayannath@cmi.ac.in

Advisor: Prof. Eknath Ghate

Visiting Students' Research Program

22nd June, 2023

Introduction

We will provide a classification of smooth irreducible representations of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ over $\overline{\mathbb{F}}_p$. We then discuss 2-dimensional local Galois representations over $\overline{\mathbb{F}}_p$. Subsequently, we state Breuil's mod p local Langlands correspondence and end by proving the classification theorem.

Notations

$G = \mathrm{GL}_2(\mathbb{Q}_p)$; $K = \mathrm{GL}_2(\mathbb{Z}_p)$; $\overline{B} = \{$lower triangular matrices in $G\}$; η, χ or χ_i always denote smooth characters $\mathbb{Q}_p^\times \to \overline{\mathbb{F}}_p$; V always denotes a weight; $\mathcal{H}_G V := \mathrm{End}_G(\mathrm{ind}_K^G V)$ is the Hecke algebra; Ind denotes smooth induction; ind denotes compact induction.

Admissibility

In what follows, there is a technical notion called "admissibility", but we will ignore it for the sake of clarity and because all our G-representations are admissible anyway.

Classification of irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

Theorem 1 (Barthel-Livné)

Every irreducible G-representation falls into one of the following disjoint families:

1. *principal series:* $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$, $\chi_1 \neq \chi_2$,
2. *smooth characters:* $\chi \circ \det$,
3. *twists of Steinberg:* $\mathrm{St} \otimes (\chi \circ \det) \stackrel{def}{=} (\mathrm{Ind}_B^G(\chi \otimes \chi))/(\chi \circ \det)$,
4. *supersingular representations.*

Thanks to Breuil, we have the following characterisation of supersingular representations which we state without proof.

Theorem 2 (Breuil)

The irreducible supersingular representations of G are exactly

$$\frac{\mathrm{ind}_{K\mathbb{Q}_p^\times}^G \mathrm{Sym}^r \overline{\mathbb{F}}_p^2}{(T_1)} \otimes (\eta \circ \det)$$

Fundamental characters

- For ease of notation let us denote $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ by G_F. Also, $\mathbb{Q}_{p^2}$ be the unique unramified degree-2 extension of $\mathbb{Q}_p$.
- Let $g \in G_{\mathbb{Q}_{p^2}}$. Set $\pi_2 = \sqrt[p^2-1]{-p}$. **Serre's level 2 fundamental character** ω_2 is given by composing the map

$$g \mapsto g(\pi_2)/\pi_2,$$

  which takes values in μ_{p^2-1}, with the isomorphism $\mu_{p^2-1} \xrightarrow{\sim} \mathbb{F}_{p^2}^\times$ (inverse of Teichmüller lift).
- We also have the **mod p cyclotomic character** ω which can be defined as ω_2^{p+1}. It takes values in $\mathbb{F}_p^\times$.

Galois representations

- Put $\sigma = \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^h$. It can be easily checked that the determinant of $\sigma|_{G_{\mathbb{Q}_{p^2}}}$ is $(\omega_2^h)^{p+1} = \omega^h$.
- We may twist σ by an unramified character so that its determinant on $G_{\mathbb{Q}_p}$ is ω^h.
- Denote the resulting representation by $\mathrm{Ind}\, \omega_2^h$.

Theorem 3

All irreducible 2-dimensional continuous representations of $G_{\mathbb{Q}_p}$ over $\overline{\mathbb{F}}_p$ are of the form $\mathrm{Ind}\, \omega_2^h \otimes \lambda_a$, where $p + 1 \nmid h$, and λ_a is an unramified character mapping the Frobenius to a^{-1}.

Mod p local Langlands correspondence

Theorem 4 (Breuil)

There exists an explicit bijection

$$\left\{ \begin{array}{c} \textit{irreducible (admissible)} \\ \textit{supersingular representations of} \\ \mathrm{GL}_2(\mathbb{Q}_p) \textit{ over } \overline{\mathbb{F}}_p \textit{ upto isomorphism.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{irreducible continuous} \\ \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p) \\ \textit{upto isomorphism.} \end{array} \right\}$$

Idea of proof. The bijection is the following:

$$\frac{\mathrm{ind}_{K\mathbb{Q}_p^\times}^{G} \mathrm{Sym}^r \overline{\mathbb{F}}_p^2}{(T_1)} \otimes (\eta \circ \det) \longleftrightarrow \mathrm{Ind}\, \omega_2^{r+1} \otimes \eta,$$

where on the RHS, η acts on $\overline{\mathbb{F}}_p$ via local class field theory. □

Proof of Theorem 1

Let π be an irreducible representation and let V be a weight of π.

- The (finite-dimensional!) weight space $\mathrm{Hom}_K(V, \pi|_K)$ contains a common Hecke eigenvector $f\colon V \hookrightarrow \pi|_K$ with eigenvalues given by some algebra homomorphism $\chi'\colon \mathcal{H}_G V \to \overline{\mathbb{F}}_p$.
- If $\chi'(T_1) = 0$ for all V, then π is *defined* to be supersingular. So, let us assume $\chi'(T_1) \neq 0$.
- By Frobenius reciprocity applied to f, we get a nonzero (and hence, surjective) map $\mathrm{ind}_K^G V \twoheadrightarrow \pi$, which factors as $\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G V} \chi' \twoheadrightarrow \pi$.

There are now several possibilities...

Case 1 If $\dim V > 1$, then one can prove that

$$\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G V} \chi' \cong \mathrm{Ind}_B^G \chi_1 \otimes \chi_2$$

for some choice of characters χ_1, χ_2 (Atharva's talk). Hence, π is either an irreducible principal series or a twist of Steinberg.

Proof of Theorem 1

Case 2 If $\dim V = 1$ and $\chi'(T_1^2 - T_2) \neq 0$, then we have

$$\operatorname{ind}_K^G V \otimes_{\mathcal{H}_G V} \chi' \cong \operatorname{ind}_K^G V' \otimes_{\mathcal{H}_G V} \chi' \qquad (\star)$$

for some p-dimensional weight V'.

- Using classification of weights (Sudharshan's talk), one can find maps

$$\varphi^- \colon \operatorname{ind}_K^G V' \to \operatorname{ind}_K^G V, \qquad \varphi^+ \colon \operatorname{ind}_K^G V \to \operatorname{ind}_K^G V'.$$

- Identifying $\mathcal{H}_G V \cong \overline{\mathbb{F}}_p[T_1, T_2, T_2^{-1}] \cong \mathcal{H}_G V'$, we can view $\varphi^- \circ \varphi^+$ and $\varphi^+ \circ \varphi^-$ as algebra endomorphisms of $\overline{\mathbb{F}}_p[T_1, T_2, T_2^{-1}]$.
- It can be verified by a routine computation that

$$\varphi^+ \circ \varphi^- = \varphi^- \circ \varphi^+ = T_1^2 - T_2.$$

- We have

$$(\varphi^+ \circ \varphi^-) \otimes_{\overline{\mathbb{F}}_p[T_1, T_2, T_2^{-1}]} \chi' = \chi'(T_1^2 - T_2) \neq 0.$$

  Hence, their compositions act invertibly by the nonzero scalar $\chi'(T_1^2 - T_2)$.

One can now proceed as in the previous case.

Proof of Theorem 1

Case 3 If $\dim V = 1$ and $\chi'(T_1^2 - T_2) = 0$, then we may assume $V = \mathbb{1}_K$ simply by twisting by a character of the form $\eta \circ \det$. We may further arrange $\chi'(T_1) = \chi'(T_2) = 1$ by ensuring $\eta(p) = \chi'(T_1)$. One can prove that there is an exact sequence

$$0 \longrightarrow \mathrm{St} \longrightarrow \frac{\mathrm{ind}_K^G \mathbb{1}_K}{(T_1 - 1, T_2 - 1)} \xrightarrow[\text{in Bruhat-Tits tree}]{\text{sum all nodes}} \mathbb{1}_G \longrightarrow 0.$$

The middle term is the same as $\mathrm{ind}_K^G \mathbb{1}_K \otimes_{\mathcal{H}_G V} \chi'$. It follows that $\pi \otimes (\eta \circ \det)$ is the trivial character. Twisting back, π is a character of the desired form.

Finally, one shows that the four families discussed above are disjoint by doing an analysis on their weights and Hecke eigenvalues. □

References

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